point of division on the diameter pass a line, as DE (E being the further intersection of the line and the circle). The chord AE will give one of the divisions desired.

4. Dr. Murnaghan considered the problem of finding the limiting point reached when one starts at any point on an ellipse and joins this point to a focus, and joins the second intersection of this chord and the ellipse to the other focus, and so on, *cf.* Problem **3167** of this MONTHLY. The mapping $z=t+(a^2/t)$ sends the unit circle $|t|=1$ into an ellipse and if the various points on the unit circle corresponding to the points on the ellipse are, in order, t_1, t_1', t_2, t_2', etc. it is readily found that

$$\frac{t_2+1}{t_2-1}=q^2\frac{t_1+1}{t_1-1} \text{ where } |q|<1.$$

Hence

$$\frac{t_n+1}{t_n-1}=q^{2n}\frac{t_1+1}{t_1-1} \text{ so that } \lim_{n\to\infty} t_n=-1.$$

The limiting point on the ellipse is an end of the major axis.

5. Professor Bingley's paper appears in this MONTHLY (*1926*, 418).

6. The representation of point events (an event being given by two coordinates x, y and the time t) with the aid of directed circles in the plane was discussed. A continuous series of events, *i. e.*, a particle in motion, is represented by a family of such circles; collision between two particles—by two families which have one circle in common. The "distance" or "interval" between two events is represented by the distance between the corresponding circles measured along the common tangent. This leads in a natural way to the formula for the interval with one negative and two positive squares.

7. Distribution and nativity maps aid in location of school sites and Americanization program respectively. Comparison of statistics showing percent of each mark given by each teacher aids both teacher and rating officials. Non-conformity to probability curve shown in (1) percents of marks in different subjects, (2) in averages of graduates (computed for college certification), (3) in averages of undergraduates (computed for admission to the honor society) and (4) in the IQ's of entrants, makes marking on basis of normal curve unjust. Insignificant correlation is found between IQ's and achievement ($r=.38$) and between absence and failure.

8. Mr. Robinson demonstrated an instrument that would draw a certain class of curves, namely those which have the property that when a circle rolls upon them a point connected with the circle will trace out a circle in the plane over which the rolling circle moves. These curves are in general given analytically by elliptic integrals. Adjustments can be made on the instru-

ment so that the graph of any particular elliptic integral of the first or third kind can be found.

9. Professor Tyler discussed the properties and applications of the operator e^{PD_x}, where P is a function of x and D_x is the differential operator d/dx. This operator e^{PD_x} has the property of producing a substitution. Thus $e^{PD_x} \cdot f(x) = f(\lambda)$, where λ is a function of x, for example, $e^{\log_x y x D_x} \cdot f(x) = f(xy)$. The relation between P and λ is given by the functional equation

$$\int \frac{d\lambda}{P(\lambda)} - \int \frac{dx}{P(x)} = 1.$$

Various illustrations were given as to how this operator could be used to solve functional equations.

10. The three difference equations

$$F(x+1) = x^x F(x),\ G(x+1) = \Gamma(x)G(x),\ f(x+1) = \sin xf(x),$$

can be solved very elegantly by passing to a sufficiently high derivative of the logarithms of the equations. This device is also sometimes used in the derivation of the gamma function itself. The F function was first derived by Kinkelin in *Crelle's Journal*, vol. 57; the G function was first derived by Alexeiewsky and later by Barnes in the *Quarterly Journal*, vol. 31. Kinkelin used the sumbol G for his function but the symbol F was used in this paper to distinguish it from the Barnes function. Besides these three functions, the paper included a fourth function defined by the difference equation $Z(x+1) = F(x)Z(x)$, in which the F function is the Kinkelin function.

11. The use of lists of extra credit problems in connection with a course in analytic geometry was discussed. Miss Arnold has used such lists of problems very successfully and has not sectionized her classes.

12. Professor Root discussed the problem of finding the frequency of free torsional vibrations of a shaft with a reciprocating engine at one end and a rotating mass at the other end. The vibrations of a shaft with two rotating bodies rigidly attached are given by linear differential equations, but the reciprocating parts of the engine introduce the square of the first derivative and bring into the coefficients periodic functions of the dependent variable. For certain multicylinder engines the lower harmonics of the periodic coefficients balance out, and the problem is solved approximately by linear differential equations with constant coefficients.

The following officers were elected for the coming year: Chairman, J. A. BULLARD, U. S. Naval Academy; Secretary-Treasurer, J. R. MUSSELMAN, Johns Hopkins University; Members of Executive Committee, E. C. PHILLIPS, Georgetown University, and T. H. TALIAFERRO, University of Maryland.

J. A. BULLARD, *Secretary-Treasurer.*

THE ORIGIN OF THE TERM "ALGEBRA"

By SOLOMON GANDZ, Rabbi Isaac Elchanan Theological Seminary, New York City

1. A small volume might be filled with an enumeration of the various attempts to translate and explain the meaning of the word "algebra". Only a few examples, taken at random from the current reference books, need be given in order to illustrate the uncertainty and lack of clearness in the use of this mathematical term.

The word first occurs in the phrase "*al-jebr w'al muqâbalah*," this being the title of the first book of its type, written by Mohammed ibn Mûsâ al-Khowârizmî (*c.* 825). Because of this double title, all the explanations that have been advanced contain a comment upon the second word, "al-muqâbalah," as well as the first one. For example, consider the following instances:

(1) Murray's *English Dictionary*, under "Algebra":

"The redintegration or reunion of broken parts; hence the surgical treatment of fractures, bone-setting." 'Ilm al-jebr wa'l-muqâbalah" = science of redintegration and equation (opposition, comparison, collation)."

(2) *Encyclopaedia Britannica:* "Transposition and removal."

(3) Lane, *Arabic English Dictionary*, (I, 374): "Perfective addition and compensative subtraction," or "restoration and compensation."

(4) Rosen, in his edition of the *Algebra* of Mohammed al-Khowârizmî, English text, p. 3, translates: "Completion and reduction." In his notes, pp. 177-186. there is an extended discussion of the subject, based upon certain old Arabic authorities, and reading about as follows: "Al-jabr = restoration of a broken bone; in mathematical language: removal of the negative quantity. Al-muqâbalah = to put two things face to face, to confront or compare; in mathematical language, to reduce the equation by removal of two positive quantities which are equal on both sides." This is also the way in which al-Khowârizmî uses the two expressions. For the restoration of a fraction by means of a multiplication al-Khowârizmî uses the verb "ikmâl," and for the reduction by means of a division he uses the verb "radd."

(5) Cantor, *Vorlesungen zur Geschichte der Mathematik* (I (1), 619, 620), quotes the usual Latin translation: "restauratio et oppositio," which he renders in the German: "Herstellung und Gegenüberstellung." From the fact that al-Khowârizmî used these words without any explanation, this eminent historian of mathematics draws the conclusion that they must have been traditional terms, well known a long time before they appeared in the book itself.[1]

[1] This conclusion is reasonable as well as logical and will serve to support the writer's own suggestion as given later in this paper.

(6) The discussion given by Professor Smith in his *History of Mathematics* (II. p. 386–392) is also very instructive. He quotes the explanation given by Behâ Eddîn (1600) in his *Kholâsat al-Hisâb*[1], and interprets it very clearly as follows: "*al jabr* has as the fundamental idea the transposition of a negative quantity, and *muqâbalah* the transposition of a positive quantity and the simplification of each number." Smith also gives an interesting list of the distortions of the terms *al-jabr w'al-muqâbalah* by the Latin translators (p. 390), and is the first to suggest[2] the comparative use of the other Semitic languages for the explanation of *al jabr* (p. 388, note 2).

(7) Among the Arabic authorities, al-Karkhî (*c.* 1020), offers an explanation which differs from the traditional one, but his idea has had two quite different interpretations. Woepcke[3], referring to his *Fakhrî*, states that *al-jabr* is the removal of the negative and positive members of the equation, while *al-muqâbalah* simply means: to set up the equation. Ruska,[4] however, referring to the definition given in the book *Al Kâfî*, relates that the operations of removing the negative quantities and the fractions is meant by *al-jabr*, while the removal of the equal positive quantities is indicated by *al-muqâbalah*. Since the Arabic texts of *Fakhrî* and *Al-Kâfî* are not yet edited and available, the writer is not able to decide which relation is the correct one.

(8) Later Arabic authorities, of the 14th and 15th centuries, introduce the terms *al-jabr w'al-ḥaṭṭ* instead of *al-jabr w'al-muqâbalah*. *Al-jabr* means completion, *i.e.*, the removal of the negative quantities and fractions by the two operations $a+x=b$, $a \cdot x=b$; while *al-ḥaṭṭ* means the reduction, *i.e.*, the removal of the equal positive members, and of the multiplying factors by the two operations[5] $a-x=b$, $(a/x)=b$.

(9) Finally Ruska, in his *Zur ältesten arabischen Algebra*, pp. 5–14, devotes a whole chapter to this question. He realizes a great many of the difficulties, gives a full account of the literature on the subject, and also sets forth the usage of those terms by the old Arabic mathematicians. His own translation of the expression,—*Ergänzung und Ausgleichung* (completion and compensation) does not, however, bring us any nearer to the solution of the problem.

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[1] Arabic text, ed. Nesselman, p. 41.

[2] This paper owes its inception to that suggestion.

[3] See *Extrait du Fakhrî*, pp. 7, 63-64; compare also Smith, *History*, I, 283–284.

[4] *Zur ältesten arabischen Algebra*, Heidelberg 1917, pp. 13–14; compare also Hochheim, *Al-Kâfî fîl Ḥisab*, III, p. 10.

[5] See Carra de Vaux in *Bibliotheca Mathematica*, 1897, pp. 1–2; Ruska, *l.c.*, pp. 11–12. A similar explanation of *al-jabr* is also given by Ibn Khaldûn (1332–1406), in his *Muqaddamah*, ed. Beirut, 1886, p. 422; see Ruska, *l.c.*, p. 14.

2. Before proposing his own solution the writer wishes to set forth all of the difficulties, as follows:

(1) *Jabara* apparently has no etymology in the Arabic language; "to set a broken bone" is a derived meaning. We are at once confronted by the question, Why should we use an artificial surgical term for a mathematical operation, when there are such good plain words as *zâda* and *tamma* for the operation of addition and completion?

(2) *Muqâbalah* means "putting face to face, confronting, equation," and the question arises as to the reason for giving to it the meaning of the special operation of removing the equal positive members.

(3) Why should the names of these two operations give the name for the whole science of equations?

(4) There are still remnants in the mathematical literature suggesting that in olden times the term *al-jabr* alone was used for the science of equations, and the term *al-jabriyyun* was taken for the masters of algebra.[1] On the other hand the term *al-muqâbalah* alone, according to its real meaning of "putting face to face, confronting, equation," seems to be the most appropriate name for equations in general. With these difficulties in mind, the writer undertook to search out the real meaning of *jabara* in the related Semitic languages. Now the Assyrian name *gabrû-maḫâru* means to be equal, to correspond, to confront, or to put two things face to face; see Delitzsch, *Assyrisches Handwörterbuch*, under *gabru* and *maḫâru*, pp. 193, 401, and Muss-Arnolt, *Assyrian Dictionary*, under *gabru and maxaru*,[2] pp. 210, 525. From the first of these we have the etymology of the Hebrew *geber* and *gibbôr*. *Geber* is the mature man leaving the state of boyhood and being *equal in rank and value* to the other men of the assembly or army. *Gibbôr* is the hero who is strong enough *to fight and overcome his equals and rivals* in the hostile army. *Gabara=jabara*, in its original Assyrian meaning, is therefore the corresponding name for the Arabic *qâbala* (verbal noun *muqâbalah*), and an appropriate name for equations in general.

3. The Egyptians[3] knew and wrote books on algebra as early as 1600 B.C., and it would be very strange if the Assyrians, having the same level of culture as the Egyptians and having close political and economic relations with them, were quite ignorant of this art. *Gabr* must have been the original Assyrian form of the word. The Arabs received this ancient science, with its original Assyrian name (in Arabic pronunciation *al-jabr*) from the Aramaeans

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[1] See Omar al-Khayyâm, ed. Woepcke, Arabic text, pp. 2, line 7; p. 4, l. 14; p. 5, ll. 6, 17; p. 6, l. 8; p. 7, l. 7, and Ruska *l.c.*, p. 13, seq.; p. 29, note 1.

[2] The latter transcribes *maxaru* instead of *maḫaru*.

[3] Smith, *History of Mathematics*, I, 47 seq.; II, 370.

and Syrians, who lived on Assyrian territory, and added the Arabic name *al-muqâbalah*, which is nothing else than the literal Arabic translation of *al-jabr*. This took place many hundreds of years before Mohammed ibn Mûsâ al-Kowârizmî. Later on the real meaning of the word was forgotten, and the simple meaning seems not to have been regarded as scholarly enough for good usage. The scholastic method at that time was common in both the philosophical and the theological schools. The scholars tried to find in the Bible, the Koran, and the old philosophical texts everything but the simple, plain meaning. The same method was followed towards these two mathematical terms. The masters simply declared them to signify the first two operations of algebra, namely the removal of the negative and positive quantities, without worrying much about philological reasons. This conception of the two words was already well known at the time of Mohammed ibn Mûsâ al Khowârizmî, and the latter used the terms in the traditional way without any further explanation, as shown by both Rosen and Ruska.

For more than a thousand years this scholastic interpretation prevailed in the Arabic and European worlds. In reality, however, it would seem that the expression *'Ilm al-jabr w'al-muqâbalah* ought to be rendered simply as *Science of equations*, *al-jabr* being the Assyrian and *al-muqâbalah* the Arabic name for equation.

ON THE CORRELATION BETWEEN TWO FUNCTIONS[1]

By F. M. WEIDA, Lehigh University.

1. **Introduction.** The recent papers of P. R. Rider (1924, 227–231), Karl Pearson (1925, 70–73), and H. L. Rietz[2] suggested to me the investigation of the correlation between any two rational integral functions of a given set of uniformly distributed values on a given range.

It is the purpose of the present paper to consider the measurement of the degree of correlation between two variates X and Y when

$$X = \sum_{i=0}^{n} a_i t_i, \qquad (i = 0,1,2, \cdots, n) \tag{1}$$

and

$$Y = \sum_{j=0}^{m} b_j t_j, \qquad (j = 0,1,2, \cdots, m) \tag{2}$$

where the $a_i's$ and the $b_j's$ are real numbers, and the variate t is defined for all values t_k of t on the range of real numbers $c \leqq t_k \leqq d$.

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[1] Presented before the American Mathematical Society, New York, Jan. 1, 1926.

[2] *The Quarterly Publications of the American Statistical Association*, Sept. 1919, pp. 472–476.

In the conventional notation, let $\bar{X}$ and σ_x be the arithmetic mean and standard deviation of the values of X. Similarly, let $\bar{Y}$ and σ_y be the arithmetic mean and standard deviation of the values of Y. It will be found convenient to let

$$A_0 = \frac{a_0 - \bar{X}}{\sigma_x}, \quad A_i = \frac{a_i}{\sigma_x}, \quad B_0 = \frac{b_0 - \bar{Y}}{\sigma_y}, \quad B_j = \frac{b_j}{\sigma_y}. \tag{3}$$

Then we shall prove the following

2. **Theorem.** *The correlation coefficient between corresponding values of X and Y is given by*

$$r = \frac{1}{d-c} \sum_{s=0}^{n+m} \frac{L_s}{s+1} (d^{s+1} - c^{s+1}), \tag{4}$$

where $L_s = A_sB_0 + A_{s-1}B_1 + A_{s-2}B_2 + \cdots + A_0B_s$, when $0 \leqq s \leqq n$;

and $L_s = A_{s-k}B_k + A_{s-k-1}B_{k+1} + \cdots + A_{k-1}B_m$, when $s = n + k$; $0 < k \leqq m$.

To prove the theorem, we have almost directly from the definitions[1] of arithmetic mean and standard deviation

$$\bar{X} = \frac{1}{d-c} \int_c^d X dt, \quad (5), \qquad \bar{Y} = \frac{1}{d-c} \int_c^d Y dt, \tag{6}$$

$$\sigma_x^2 = \frac{1}{d-c} \int^d X^2 dt - \bar{X}^2, \quad (7) \quad \sigma_y^2 = \frac{1}{d-c} \int_c^d Y^2 dt - \bar{Y}^2. \tag{8}$$

By substituting the values of X and Y from (1) and (2) in (5), (6), (7), (8) and integrating, we obtain

$$\bar{X} = \frac{1}{d-c} \sum_{i=0}^{n} \frac{a_i}{i+1} (d^{i+1} - c^{i+1}), \tag{9}$$

$$\bar{Y} = \frac{1}{d-c} \sum_{j=0}^{m} \frac{b_j}{j+1} (d^{j+1} - c^{j+1}), \tag{10}$$

$$\sigma_x = \left[\frac{1}{d-c} \sum_{i=0}^{2n} \frac{m_i}{i+1} (d^{i+1} - c^{i+1}) - \bar{X}^2 \right]^{\frac{1}{2}}, \tag{11}$$

$$\sigma_y = \left[\frac{1}{d-c} \sum_{j=0}^{2m} \frac{n_j}{j+1} (d^{j+1} - c^{j+1}) - \bar{Y}^2 \right]^{\frac{1}{2}}, \tag{12}$$

[1] E. V. Huntington, MONTHLY, (1919, 426).

where m_i is the sum $\sum_{r=1}^{i+1} a_{r-1}a_{i-r+1}$ when $0 \leqq i \leqq n$. When $(n+1) \leqq i \leqq 2n$, m_i is the sum $\sum_{r=1}^{2n-i+1} a_{i-n+r-1}a_{n-r+1}$. Similarly n_j is the sum $\sum_{r=1}^{j+1} b_{r-1}b_{j-r+1}$ when $0 \leqq j \leqq m$. When $(m+1) \leqq j \leqq 2m$, n_j is the sum $\sum_{r=1}^{2m-j+1} b_{j-m+r-1}b_{m-r+1}$.

If now we introduce the notation $x=(X-\overline{X})/\sigma_x$, $y=(Y-\overline{Y})/\sigma_y$, we may write

$$x = \sum_{i=0}^{n} A_i t^i, \qquad y = \sum_{j=0}^{m} B_j t^j. \tag{13}$$

Then applying the definition that the correlation coefficient is the arithmetic mean of the products xy over the range on t from c to d, we have

$$r = \frac{1}{d-c}\int_c^d \left(\sum_{i=0}^{n} A_i t^i \sum_{j=0}^{m} B_j t^j\right)dt = \frac{1}{d-c}\int_c^d \sum_{s=0}^{s=n+m} L_s t^s dt \tag{14}$$

$$= \frac{1}{d-c}\sum_{s=0}^{n+m} \frac{L_s}{s+1}(d^{s+1}-c^{s+1}), \tag{15}$$

which was to be proved.

This form of r is useful for numerical calculation when application is to be made to particular cases.

3. **Special cases.** Let us now consider the following simple special cases, namely,

(1) $X = a_0 + a_1 t, \qquad Y = b_0 + b_1 t;$

(2) $X = a_0 + a_1 t, \qquad Y = b_0 + b_1 t + b_2 t^2;$

(3) $X = a_0 + a_1 t + a_2 t^2, \qquad Y = b_0 + b_1 t + b_2 t^2;$

and attempt a simplification of our formula (15) for r.

(1) When $X=a_0+a_1t$, $Y=b_0+b_1t$. Here $m=n=1$, and we write for (15)

$$r = \frac{1}{d-c}\left\{\frac{L_0}{1}(d-c) + \frac{L_1}{2}(d^2-c^2) + \frac{L_2}{3}(d^3-c^3)\right\}, \tag{16}$$

where $L_0 = A_0B_0$, $L_1 = A_1B_0 + A_0B_1$, and $L_2 = A_1B_1$.
From (3), (5), (6), (7), (8), we find that

$$A_0 = \frac{\mp (d+c)\sqrt{3}}{d-c}; A_1 = \frac{\pm 2\sqrt{3}}{d-c}; B_0 = \frac{\mp (d+c)\sqrt{3}}{d-c}; B_1 = \frac{\pm 2\sqrt{3}}{d-c}.$$

Hence,

$$L_0 = \frac{\pm 3(d+c)^2}{(d-c)^2}; \quad L_1 = \frac{\mp 12(d+c)}{(d-c)^2}; \quad L_2 = \frac{\pm 12}{(d-c)^2},$$

Substituting the values of L_0, L_1, L_2, just found, in equation (16) and simplifying, we find that

$$r = \pm 1. \tag{17}$$

(2) When $X = a_0 + a_1 t$, $Y = b_0 + b_1 t + b_2 t^2$. In this case $n = 1$ and $m = 2$, and we write for (15)

$$r = \frac{1}{d-c}\left\{\frac{L_0}{1}(d-c) + \frac{L_1}{2}(d^2-c^2) + \frac{L_2}{3}(d^3-c^3) + \frac{L_3}{4}(d^4-c^4)\right\}, \tag{18}$$

where $L_0 = A_0B_0$, $L_1 = A_1B_0 + A_0B_1$, $L_2 = A_1B_1 + A_0B_2$, and $L_3 = A_1B_2$.

If now we find the values of the L's by making use of (3), (5), (6), (7), (8); and if we substitute these values repectively in (18) and simplify, we find that

$$r = \sqrt{15}\,\{b_1 + b_2(d+c)\}/\{15b_1^2 + 30b_1b_2(d+c) + 4b_2^2(4d^2+7dc+4c^2)\}^{\frac{1}{2}}. \tag{19}$$

From this result, let us investigate the conditions under which $r^2 = 1$. If we square both members of equation (19) and impose the condition that $r^2 = 1$ and solve the resulting equation, we find that when $b_2 \neq 0$, $r^2 = 1$ only in the trivial case when $d = c$.

(3) When $X = a_0 + a_1 t + a_2 t^2$, $Y = b_0 + b_1 t + b_2 t^2$. Here $m = n = 2$, and we write for (15) that

$$r = \frac{1}{d-c}\left\{\frac{L_0}{1}(d-c) + \frac{L_1}{2}(d^2-c^2) + \frac{L_2}{3}(d^3-c^3) + \frac{L_3}{4}(d^4-c^4) + \frac{L_4}{5}(d^5-c^5)\right\}, \tag{20}$$

where $L_0 = A_0B_0$, $L_1 = A_1B_0 + A_0B_1$, $L_2 = A_2B_0 + A_1B_1 + A_0B_2$, $L_3 = A_2B_1 + A_1B_2$, and $L_4 = A_2B_2$.

If now we find the values of the L's by making use of (3), (5), (6), (7), (8); and if we substitute these values respectively in (20) and simplify, we find that

$$r = \frac{15a_1b_1 + 15(a_2b_1 + a_1b_2)(d+c)}{\{15a_1^2 + 30a_1a_2(d+c) + 4a_2^2(4d^2+7dc+4c^2)\}^{1/2}} + \frac{4a_2b_2(4d^2+7dc+4c^2)}{\{15b_1^2 + 30b_1b_2(d+c) + 4b_2^2(4d^2+7dc+4c^2)\}^{1/2}}. \tag{21}$$

In particular, if $a_1/a_2 = b_1/b_2$, and if a_2 and b_2 have the same sign, the vertices of our (X, t) and (Y, t) parabolas have the same t-coordinate and the para-

bolas extend in the same direction. If now we impose these conditions upon the a's and b's in (21) we readily find that

$$r=1. \tag{22}$$

Again if $a_1/a_2=b_1/b_2$, and if a_2 and b_2 have opposite signs, the vertices of our (X, t) and (Y, t) parabolas have the same t-coordinate and the parabolas extend in opposite directions. If we impose these conditions upon the a's and the b's in (21) we find that

$$r=-1. \tag{23}$$

The results $r=\pm 1$ from $a_1/a_2=b_1/b_2$ are of analytic interest because $r=\pm 1$ without regard to the distances from the vertices to the foci of the parabolas.

4. **Concluding remarks.** It is fairly obvious that the correlation we have studied may be regarded geometrically as the correlation between the ordinates of two curves given by (1) and (2) on the range $c \leqq t \leqq d$.

While the substitution of special values in (15) shows that the correlation coefficient between these ordinates may take a wide range of values varying from -1 to 1, it should perhaps be remarked that the correlation ratio[1] η_{yx} of y on x would be equal to unity when the elimination of t from (1) and (2) gives y as a single valued function of x.

In the preceding discussion, we have placed no restrictions on c or d except that c and d are real numbers. A case of theoretic interest arises when the difference between c and d is very small. We then write $d=c+\Delta c$ where Δc is very small. As $\Delta c \to 0$, the segments of the two continuous curves may be taken as strictly linear and the correlation coefficient will approach either 1, -1, or 0. An examination of (19) and (21) when $d=c+\Delta c$ and $\Delta c \to 0$ shows that $r=\pm 1$.

CONCERNING THE PROBABILITY CURVES OF N POINTS TAKEN AT RANDOM ON A STRAIGHT LINE SEGMENT OF CONSTANT LENGTH

By W. B. CHADWICK, Wilmington, Delaware.

In the theory of probability, when a point P is taken at random upon a segment of a straight line and the probability that P shall fall at a given point is constant for all points of the line, the probability that P shall fall in a segment of length Δx is said to be $\Delta x/a$, where a is the length of the original segment upon which it is certain the point must fall. If the probability is not constant for all points but is a continuous function of x, the abscissa of the corresponding

[1] *Handbook of Mathematical Statistics* by H. L. Rietz and others, (1924), pp. 129–131.

point, $y=f(x)$ is said to be the probability curve of the point P and we have the condition $\int_0^a f(x)dx=1$. In the case of one point taken at random dx/a is the probability that P will fall between x and $x+dx$ and $\int_0^a (1/a)dx=1$, so that $y=1/a$ is the probability curve of the point. When two points X, Y are taken at random upon the line AB, whose length is a, X being the point nearer A, dx/a is the probability that X is between x and $x+dx$ and dy/a that Y is between y and $y+dy$, so that $dxdy/a^2$ is the probability that X is between x and $x+dx$ and Y is between y and $y+dy$. Then $(dx/a^2)\int_x^a dy$ sums the probabilities when Y is to the right of X, so that we have $y=c(a-x)/a^2$ as the probability curve for X, where c must be determined subject to the condition that $(c/a^2)\int_0^a (a-x)dx = 1$, for although $\int_0^a\int_0^a dxdy/a^2 = 1$, by imposing the condition that X is to the left of Y, we have restricted the range of values in determining the probability curve as a function of x so that there is no reason to expect that this function integrated over the entire range will give unity as a result. Accordingly, $y=2(a-x)/a^2$ is the probability curve of X, and similarly $y=2x/a^2$ is the probability curve of Y. We are now ready to turn our attention to the general case.

LEMMA ONE. If n points are taken at random upon a straight line segment of constant length a, and P_k be the kth point from the left, then the probability curve of P_k has a zero of the $(k-1)$ th order at the origin and a zero of the $(n-k)$th order at the point $(a, 0)$, where the segment a is that portion of the x-axis between the origin and $(a, 0)$. For, summing up the probabilities of the positions of the $(n-1)$ other points, one possibility is that $(k-1)$ of them may be adjacent at the origin and that $(n-k)$ may be adjacent at the point $(a, 0)$. Hence the ordinate of the probability curve for the point P_k is zero for the $(k-1)$ adjacent points at the origin and the $(n-k)$ adjacent points at $(a, 0)$.

LEMMA TWO. The probability curve of P_k any one of the n points taken under the above hypothesis is of degree $(n-1)$. For we have:

$$\frac{dp_k}{a^n}\int_{p_{n-1}}\cdots\int_{p_{k+1}}\int_{p_{k-1}}\cdots\int_{p_1} dp_1\cdots dp_{k-1}dp_{k+1}\cdots dp_{n-1}$$

upon summing up the various probabilities, and each integration is performed with respect to at least one variable.

From these lemmas we infer that $y=c(a-x)^{n-k}\,x^{k-1}$ is the probability curve of the kth point. To determine c set $c\int_0^a (a-x)^{n-k}\,x^{k-1}\,dx=1$. By means of the Beta and Gamma functions we obtain:

$$c = \frac{n}{a^n}\,\frac{(n-1)(n-2)\cdots(n-k+1)}{(k-1)!},$$

so that the probability curve of the kth point from the left is: $y=(n/a^n)$ multiplied by the kth term in the expansion of $[(a-x)+x]^{n-1}$. The binomial expansion thus plays an important part here as in other types of probability problems.

To find the maximum point of this curve we set (dy/dx) equal to zero and find

$$x=a(k-1)/(n-1). \tag{1}$$

Note that when $k=1$ or n we still have the maximum of the first or the last curve with respect to that portion of the plane under consideration, namely, that portion within the rectangle three of whose sides are the x-axis, the y-axis, and the line $x=a$, although these are not maximum points of the curves themselves. This shows that the abscissas of the n points divide the x-axis from the origin to $(a, 0)$ into $(n-1)$ equal parts.

By substitution the corresponding y-coordinate of the maximum for the curve is found to be:

$$y=\frac{n(n-1)(n-2)\cdots(n-k+1)}{a(k-1)!\,(n-1)^{n-1}}(n-k)^{n-k}(k-1)^{k-1}=A\,. \tag{2}$$

When $k=1$ or n we have a term of the form 0^0 but since the limit of $(k-1)^{k-1}$ or $(n-k)^{n-k}$ as k approaches 1 or n respectively, is unity, our formula will still apply in these cases.

When k is less than $\frac{1}{2}n$, the ordinate of the maximum point for the $(k+1)$th curve is less than the corresponding ordinate for the kth curve. For if in (2) we let k be replaced by $k+1$, we get $y=B$, where

$$B=A\left(\frac{n-k-1}{n-k}\right)^{n-k-1}\left(\frac{k}{k-1}\right)^{k-1}.$$

Consider

$$\left(\frac{n-k-1}{n-k}\right)^{n-k-1}\cdot\left(\frac{k}{k-1}\right)^{k-1} \tag{3}$$

as a function of n and k.

When $n=2k$, we have

$$\left(\frac{k-1}{k}\right)^{k-1}\left(\frac{k}{k-1}\right)^{k-1}=1.$$

When $n>2k$, or $k<\frac{1}{2}n$, let $n=2k+\Delta n$.

Then (3) becomes

$$\left(\frac{k+\Delta n-1}{k+\Delta n}\right)^{k+\Delta n-1}\left(\frac{k}{k-1}\right)^{k-1}=\frac{1}{\left(1+\dfrac{1}{k+\Delta n-1}\right)^{k+\Delta n-1}}\left(\frac{k}{k-1}\right)^{k-1}$$

and

$$\frac{1}{\left(1+\frac{1}{k+\Delta n-1}\right)^{k+\Delta n-1}}$$

represents terms in the expansion of

$$\frac{1}{\left(1+\frac{1}{x}\right)^{x}},$$

where $(k+\Delta n-1)=x$ and x becomes infinite. This, however, is a decreasing function whose limit is $1/e$. Hence $B=A$ multiplied by a factor less than one when $k<\frac{1}{2}n$. Because, for a given value of n, the probability curves are evidently symmetrical with respect to the line $x=\frac{1}{2}a$, since the theory might have been developed starting from the right rather than the left, with no change, we infer that when $k>\frac{1}{2}n$, $B>A$. When n is odd, the lowest maximum point for any one of the n curves is for the kth curve when $k=\frac{1}{2}(n+1)$; if n is even the lowest maximum points are the two having equal ordinates where $k=\frac{1}{2}n$ and $k=\frac{1}{2}n+1$. Using the parametric equations (1) and (2) and eliminating the parameter k, we find that the n maximum points lie on the curve:

$$y=\frac{(a-x)^{\left(\frac{n-1}{a}\right)(a-x)}\,x^{\frac{(n-1)x}{a}}}{\int_0^a (a-z)^{\left(\frac{n-1}{a}\right)(a-z)}\,z^{\frac{(n-1)z}{a}}\,dz}$$

which is continuous when $-a/n-1<x<na/n-1$. This curve may easily be proved symmetric with respect to the line $x=\frac{1}{2}a$.

We shall now show that each one of the n curves intersects every other one once and only once in the region we are considering, except for intersections at the origin and at $(a, 0)$.

If we take the equations of any two curves and solve them simultaneously we shall have, after factoring out all factors of the form $(a-x)$ and x, $(a-x)^p=cx^p$, $p<n$, $c>0$. All the pth roots of this equation will be found in the determinations of the pth roots of c. When p is odd, c has only one real root and x is between 0 and a. When p is even, there are three cases:

(1) $c=1$. Here $(a-x)=\pm x$, gives the real values of x, but only the positive sign can be used.

(2) $c>1$. Here we have $(a-x)=\pm\sqrt[p]{c}x$ and the positive sign gives the only real positive value for x.

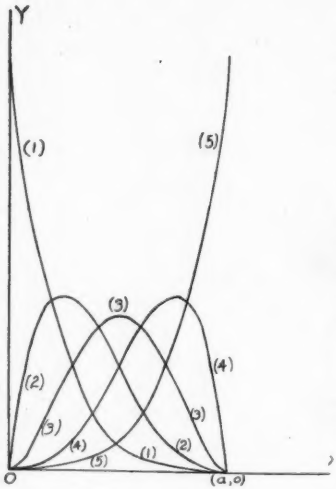
(3) $c<1$. Let $cc'=1$; then

$$(a-x)=\pm\frac{x}{\sqrt[p]{c'}};\quad x=\frac{a\sqrt[p]{c'}}{1+\sqrt[p]{c'}}\ \text{or}\ \frac{a\sqrt[p]{c'}}{\sqrt[p]{c'}-1};$$

the second of these values for x is greater than a and hence is outside the region under consideration. The ordinates of the points of intersection are, of course, positive when $0<x<a$.

As an example of the preceding consider the case $n=5$. Here the probability curves are:

$k=1$	$y=5(a-x)^4/a^5$	Max. $(0, 5/a)$
$k=2$	$y=20(a-x)^3x/a^5$	Max. $(.25a, 2.11/a)$
$k=3$	$y=30(a-x)^2x^2/a^5$	Max. $(.5a, 1.88/a)$
$k=4$	$y=20(a-x)x^3/a^5$	Max. $(.75a, 2.11/a)$
$k=5$	$y=5x^4/a^5$	Max. $(a, 5/a)$.



The curve upon which the maximum points lie is:

$$y=\frac{(a-x)^{4(a-x)/a}\,x^{4x/a}}{\int_0^a (a-z)^{4(a-x)/a}\,z^{4x/a}\,dz};\quad \frac{-a}{4}<x<\frac{5a}{4}.$$

The points of intersection of the curves are:
Curves 1 and 2—$(.2a, 2.05/a)$
Curves 1 and 3—$(.29a, 1.27/a)$
Curves 1 and 4—$(.39a, .71/a)$
Curves 1 and 5—$(.5a, .31/a)$
Curves 2 and 3—$(.4a, 1.73/a)$
Curves 2 and 4—$(.5a, 1.25/a)$.
Corresponding curves counting from the right
Curves 4 and 5—$(.8a, 2.05/a)$
Curves 3 and 5—$(.71a, 1.27/a)$
Curves 2 and 5—$(.61a, .71/a)$
Curves 1 and 5—$(.5a, .31/a)$
Curves 3 and 4—$(.6a, 1.73/a)$
Curves 2 and 4—$(.5a, 1.25/a)$.

The figure illustrates clearly the symmetry of the curves and of their points of intersection. The area under each curve, of course, is unity.

Finally it may be remarked that if, to find the probability curve of a certain point we sum the probabilities by means of $(n-1)$ integrals, allowing the position of each point to vary between that of the point preceding and that of the point following, the constant determined so that the area under the curve shall be unity, is $n!$, corresponding to the $n!$ permutations of the n points. As an example let $n=7$, $k=3$.

$$\begin{array}{c}(0,0)\qquad\qquad(a,0)\\ \overline{\qquad\qquad\qquad\qquad}\\ X\ Y\ Z\ T\ U\ V\ W\end{array}$$

$$\frac{dz}{a^7}\int_z^a\int_z^w\int_z^v\int_z^u\int_0^z\int_0^y dx\,dy\,dt\,du\,dv\,dw=(a-z)^4z^3\frac{dz}{48a^7},$$

and upon multiplying by $7!=5040$, we obtain $(105/a^7)\ (a-z)^4\ z^2\ dz$, as we may get also by our formula.

Similarly

$$\frac{dz}{a^7}\int_z^a\int_z^a\int_z^a\int_z^a\int_0^z\int_0^z dx\,dy\,dt\,du\,dv\,dw,$$

which sums the probabilities when X and Y are allowed to take any position between the origin and Z, and T, U, V, W, between Z and $(a, 0)$, gives: $(a-z)^4\ z^3\ dz/a^7$ and the required constant $c=105$ may be obtained by multiplying by $15={}_{n-1}C_{k-1}$ the number of ways in which the points may occupy the $(n-1)$ positions by crossing from left to right of Z or vice-versa, and by 7, the number of points. This last method clearly shows the existence and order of the zeros at the origin and at $(a, 0)$.

EARLY LITERARY EVIDENCE OF THE USE OF THE ZERO IN INDIA

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During the last few decades there have been many earnest attempts to find the time and country of invention of the zero and the place-value for the decimal system of numeration. Though some historians of mathematics have suggested the name of one or more of the nations that once flourished on the rich soil of Mesopotamia, the world is gradually adopting the view that the credit of the invention is entirely due to the Hindus.[1] Such was also the belief of the mediaeval and Renaissance writers of Europe. It is noteworthy that though the modern numerals which are in use all over the civilized world are commonly called "Arabic," and though some influential ancient writers pleaded with fervor in favor of Arabia, the Arabs themselves were of opinion that the invention was of Hindu origin. About the exact time of the invention there have been put forth different opinions. Most of the epigraphical instances so far found in India have been considered more or less as forgeries by some writers. Kaye[2] has been the most eloquent exponent of this group. His attitude will be known from the following quotation from this article:

> On palaeographic grounds we are forced to fix the 9th century A.D. as the earliest period in which the modern place-value system of notation may have been in use in India. This earliest period depends upon one inscription only. If this inscription, on further light being thrown upon it, prove unreliable

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[1] *Vide* Smith's introductory note to Ginsburg's article, "New Light on our numerals," *Bull. Amer. Math. Soc.*, vol 23(1917); also compare Smith and Karpinski, *The Hindu-Arabic Numerals* (Boston, 1911), Smith, *History of Mathematics*, ii, p. 64.

[2] Kaye, "Notes on Indian Mathematics—Arithmetical Notation," *Jour. Asiat. Soc. Bengal, vol. 3* (1907) p. 487.

(as it possibly will), then we shall have to fix the tenth century as the earliest period. Even for the tenth century there is not an excessive amount of good evidence, and it is within the bounds of possibility that we may have finally to turn to the eleventh century for evidence of the use of our modern system in India.

On the other hand authorities on Indian epigraphy like Bühler, V. A. Smith, Bhandarkar, and others bear testimony to the genuineness of some of the inscriptions at least and the earliest of them recording numerals with place-value is dated 595 A. D.[1] Among the writers on literary evidence, Haraprasad Shastri is of opinion that the Indians knew the place-value system of decimal numeration at least in the early centuries of the Christian era.[2] Thibaut puts the date at 590 A.D.[3] In a note recently contributed to the MONTHLY, (*1926* 220–221), I have quoted two almost identical passages from philosophical writings of the sixth and seventh centuries which expressly refer to the place value as an illustration of a form of philosophical category. I have there suggested that the higher limit of the time of invention of the place-value system of decimal numeration in India must be fixed at the fifth century. This, however, does not settle as a matter of course the date of invention of the zero as a numeral. Strictly speaking these illustrations do not mention whether the place value was indicated in writing numbers or was being indicated on the abacus. In other words it may be asked whether it will be safe to conclude that the zero was known in that early age.[4] No doubt there is the very significant fact that until now no mention of the existence of the abacus, direct or indirect, has been traced in any Indian literature so that it may be taken without any fear of contradiction that if an abacus was ever in use among the learned men of India, it was discarded long ago.[5] Hence it follows as a matter of course that, in the quotations referred to, the place value was being indicated with the help of the zero. But direct instances of the use of the zero from any early writing will be at once decisive and hence all the more welcome.

It must be remarked that almost all the literary instances should be interpreted as referring to the concept of the zero as a numeral, which must

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[1] *Epigraphia Indica*, ii, p. 29.

[2] Haraprasad Shastri, *Presidential Address to the Sanskrit and Prakrit Section, Proc. & Trans. of the Second Oriental Conference*, Calcutta (1922).

[3] Thibaut, *Astronomie, Astrologie, und Mathematik*, Strassburg, 1899.

[4] In fact the question has been put to the writer in a personal letter from Professor D. E. Smith.

[5] Fleet's interpretation of the word "ganitra" occurring in a writing of the first century A.D. as meaning "an instrument of reckoning" (Fleet, "The use of the abacus in India", *Journ. Roy. Asiat. Soc.*, 1911), cannot be taken as an instance of the use of the abacus as an instrument for arithmetical calculation. It has been stated that the man who was "expert in ganitra" was an astrologer and we know how even in modern times, the Indian astrologers are in the habit of 'casting the stars' in the form of a diagram, suggested probably by the zodiacal circle, which has no connection whatever with the mathematical calculations necessary for the purpose.

not be confounded with the symbol or the form of the symbol for the zero. We know for certain that the form of the symbol for zero has undergone changes in India from a mere dot to a small circle.[1] Who knows that there have not been other changes or that there were not other forms? Furthermore, any illustration of the form, to speak from our present knowledge, will have the difficulty that it might be explained away as meaning vacuity or disappearance. Even an instance like the one in Subandu's *Vāsabadattā*[2] (*c.* 620 A.D.) or the one in Śrīharsa's *Naiṣada-Carita*[3], where the zero has been expressly stated to be *śūnyabindy*[4] need not be considered too seriously as a decisive factor for the form of the zero symbol, being a mere point or dot. There are numerous instances in Indian literature[5] where *bindu* has been used in the sense of a small particle, as of water, oil, or gold, or of a comparatively larger spot, as in *bindu-mrga* (spotted deer) and *bindu-citra* (lit., spot-marked). Again there are positive palaeographical evidences that centuries before the time of Śrīharṣa (*c.* 12th century) who calls the zero a *śūnya-bindu*, the zero was represented in India by a small circle, as in the inscriptions of Bhojadeva[6] and of Mahīpāl.[7] All that can be safely inferred from the literary concept of the zero as a numeral is that there must have been a symbol and any further inference will not be always free from doubt.

Now the required instances are furnished by Varāhamihira's *Pañca-siddhāntikā* (505 A. D.). There he incidentally states two fundamental arithmetical operations by the zero (*kha, śūnya, ambara*), *viz.* addition and subtraction, in more than one place, *e.g.*, iii. 2, 17; iv. 7, 8, 11, 12; xviii. 35, 44, 45, 48, 51—the Roman figures indicating the chapters and the Hindu figures the verses. It is rightly stated that the value of a quantity does not change when zero is added to or subtracted from it. The true significance is not always evident in Thibaut and Dvivedi's translation, and in fairness to the Sanskrit commentator it must also be stated that he is more expressive. Hence we propose to give a fresh literal translation of the ślokas (omitting unnecessary sentences), their more complete translation being given in the footnotes.

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[1] Bühler, *On the origin of the Indian Alphabet*, p. 53, note; *cf.* the dot of the Bakshāli Mss and the small circle of the Gwalior inscription, dated Samvat 933 (876 A.D.).

[2] Hall's edition p. 181.

[3] Canto I, stanza 21.

[4] Varāhamihira has also one equivalent for zero as *bindu* not *śūnyabindu*; vide his *Pañca-siddhāntikā*, edited with English translation and Sanskrit commentary by Thibaut and Dvivedi, Benares, 1899, Chapter iv, śloka 7.

[5] For example see *Atharvaveda*, ix. i. 21; x. 10. 19; xii. 3. 29.

[6] The zero is found in two inscriptions of Bhojadeva: the first is dated 870 A.D. (Ojha, *Prāchin Lipi Mālā*, p. 127); and the second is dated 876 A.D. (Hill, *The Early Use of Numerals in Europe*, 1915).

[7] *Vide* the Asni inscription of 917 A.D., *Indian Antiquary*, vol. 16, p. 174.

Translation

iii.2[a] Corresponding to the signs of the anomaly, the numbers of minutes to be deducted (from the sun's mean longitude) are eleven; eight sixes; seventy minus one; and (that) *plus zero* [*kha*]; nine sixes; and five squared.

iii.17 "The daily motion of the sun amounts to sixty (minutes) minus three, three, three, three, two, one; plus one, one, one, one; *minus naught* [*kha*], one, in turn." [Thibaut and Dvivedi].

iv.7[b] (To which *to be added* in succession) fifty plus one; five eights; five squared; four; thirty increased by four; fifty-six; five; *zero* [*śūnya*].

iv.8[c] The signs in Taurus are six; thirteen; nineteen; three eights; again thirty *added with zero* [*ambara*], five, nine, thirteen (in succession) minutes.

iv.11[d] The seconds are sixty diminished by eighteen, three, eighteen; zero [*śūnya*]; fifty less by three; four; fifty minus one; five.

vi. 12[e] In Aries the minutes are seven, in the last sign six; in Taurus six (repeated) thrice; five (repeated) thrice; four; four; in Gemini they are three, two, one, zero [*śūnya*] (each repeated) two times.

xviii.35 "Th!rty-six *increased by* two; three, nine, twelve, nine, three, *zero* [*śūnya*] are the days. The motion in the eighth course is the same as in the seventh." [Thibaut and Dvivedi].

xviii.44[f] In Gemini (in) twice ten *plus* five squared, *zero* [*kha*], six, and three cubed (days, he passes through) half-hundred minus two; fourteeen; three cubed; and five eights (degrees).

xviii.45[g] In Cancer, (in days) four, one, three and four, multiplied by ten and *increased by* two, eight, *zero* [*śūnya*], six (respectively), (he passes through) five squared increased by one, halved, increased by one; and five squared degrees. Or symbolically:

days: $4\times10+2$; $1\times10+8$; $3\times10+0$; $4\times10+6$;
degrees: $5\times5+1$; $5\times5\times\frac{1}{2}$; $5\times5+1$; 5×5.

xviii.48[h] (In Libra, in days) twenty increased by one; *decreased by zero* [*kha*], ten, three, and then (the results) multiplied by two, (he passes through) the number of degrees less by three, eight; more by one, thirty (than the number of days, respectively). Or symbolically:

days: $(20+1)2$; $(20+0)2$; $(20+10)2$; $(20+3)2$;
degrees: $(20+1)2-3$; $(20-0)2-8$; $(20-10)2+1$; $(20-3)2+30$.

xviii.51[i] In Capricorn, (in days) twice ten *plus zero* [*kha*]; minus seven; increased by eighteen and twelve; (he passes through) the number of degrees less by one; more by one, one, twenty-six (than the number of days respectively). Or symbolically:

days: $2\times10+0$; $2\times10-7$; $2\times10+18$; $2\times10+12$;
degrees: $2\times10+0-1$; $2\times10-7+1$; $2\times10+18+1$; $2\times10+12+26$.

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[a] "Corresponding to the signs of the anomaly we have the following (aggregates of) minutes which have to be deducted or added to (from the sun's mean longitude); viz: 11, 48, 69, 69, 54, 25."

[b] "(To which to be added in succession) 51; 40; 25; 4; 34; 56; 5; 0."

[c] "The signs in Taurus are 6; 13; 19; 24; 30; 35; 39; 43 minutes."

[d] "The seconds are 42; 57; 42; 0; 47; 4; 49; 5."

[e] "In Aries the minutes are 7, in the last sign 6; in Taurus they are 6; 6; 6; 5; 5; 5; 4; 4; in Gemini they are 3; 3; 2; 2; 1; 1; 0; 0;"

[f] "In Gemini he passes in forty-five days through forty-eight degrees; in twenty days through fourteen degrees; in twenty-six days through twenty-seven degrees; in forty-seven days through forty degrees."

[g] "In Cancer he passes in forty-two days through twenty-six degrees; in eighteen days through twelve and a half degrees; in thirty days through twenty-six degrees; in forty-six days through twenty-five degrees."

[h] "(In Libra) he passes in forty-two days through thirty-seven [?-nine] degrees; in forty days through thirty-two degrees; in thirty-four days through sixty four degrees."

[i] "In Capricorn he passes in twenty days through nineteen degrees; in thirteen days through fourteen degrees, in thirty-eight days through thirty-nine degrees; in thirty-two days through fifty-eight degrees".

It is noteworthy that all the above verses are from those chapters of *Pañca-siddhāntikā* where are summarised the teachings of the *Pauliśa-siddhānta*. Thibaut remarks: "Varāhamihira has in no case obliterated the characteristic features of the Siddhantas he has to deal with, and whatever distinguishes those works from one another in the test of the *Pañca-siddhāntikā* really distinguished them in their original form."[1] Is it therefore safe to state that the method was due to the original *Pauliśa-siddhānta*? Unfortunately no means of verification of this conjecture has been left to us in the present age, the book not being now extant. It was known to Brahmagupta (*c.* 628 A.D.), to Bhaṭṭotpala (966 A.D.), and even as late as in the eleventh century to Alberuni. There are however numerous quotations in Bhaṭṭotpala's excellent commentary on Varāhamihira's *Bṛhat-samhitā*[2] from an "original Puliśa-siddhānta" and probably also from a different edition of the same work. There we find the use of the word numerals with the place value and the zero. In any case this conjecture will lead us to the conclusion that the zero was known in India before 400 A.D., for that is the higher limit for the time of composition of the original *Pauliśa-siddhānta* as set by Thibaut.[3] It may further be stated in passing, that Whitney believed that at the time of Āryabhata (476 A.D.), that Hindus had "invented their system of signs employed in decimal notation."[4]

An earlier instance of the use of the zero is found in *Chandaḥ-sūtra* of Piṅgala or Piṅgalanāga as he is otherwise called.[5] It is a manual of Vedic metres. One section of the book deals with the problem of determination of the number of possible variations for a metre with a given number of syllables. There have been formulated definite rules for the purpose of calculations and it will be interesting and rather surprising to know that the method yields a huge number of variations, such as $20{,}282{,}388 \times 10^{24}$ and even more.[6] It is in

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[1] *Pañca-siddhāntikā*, loc. cit.; Introduction, p. xvii; *cf.* p. xvi.

[2] Varāhamihira, *Bṛhat-saṃhitā*; edited by Kern, Calcutta (1895) and translated by him in *Journ. Roy. Asiat. Soc.*, (1870–75); Bhaṭṭotpala's Commentary edited by Dvivedi, Benares (1895).

[3] *Loc. cit.* p. lx. As much has been said in favor of as against the supposed identity of Puliśa, the author of the *Pauliśa-siddhānta* and an unknown Greek astronomer. But this much is absolutely certain: that this conjecture, even if it proves to be true, will not deprive the Hindus of their originality in the invention of the zero and the place-value for the decimal system of notation. For until now nothing has been discovered in the literature of the Greeks or of any other ancient nations which would justify their conception of the zero as a numeral. On the other hand Kern says that "*Pauliśa-siddhānta*, judging from the quotations, and rather numerous ones is so thoroughly Hinduised that few or no traces of its Greek origin are left." *Bṛhat-saṃhitā*, *loc. cit.*, Introduction, p. 49.

[4] *Journ. Amer. Orient. Soc.*, vol 6, p. 563 fn. The date 476 is that of Āryabhaṭa's birth.

[5] Piṅgala, *Chandaḥ-sūtra*: the text with the commentary of Halāyudha edited by Visvanath Sastri in *Bibliotheca Indica* (1871-74); also edited and commented by Weber, *Indische Studien*, viii (1863).

[6] Colebrooke, *Miscellaneous Essays*, ii (1873) p. 88.

this connection that Piṅgala has used the word *śūnya* in two successive sūtras.[1] These two sūtras, in fact the whole of the manual, are found restated in the Agni-Purāṇa.[2] There is no doubt that, by *śūnya*, Piṅgala was referring, not to the mere concept of nothingness, but to a definite symbol whose concept is akin to the concept of our zero numeral. Halāyudha, the earliest commentatator of *Chandaḥ-sūtra*, has explained the sūtras fully and has adduced illustrative examples as well. He is of opinion that, in the sūtras referred to, *śūnya* denotes the zero, and he has been supported by Weber who remarks that there cannot be any doubt about that.[3] Having no very weighty arguments or facts to suggest to the contrary, we cannot help but accept it. Now Piṅgala lived about the middle of the second century B.C.[4], and his book was popular amongst the Brāhmaṇas, the Vedic scholars of the age.[5] Hence it has to be admitted that the zero was known to the Brāhmaṇas of India in the second century B.C. The date of invention, it is highly probable, preceded this by a century or two at least.

It will be noticed that evidence from writings of authors posterior to Varāhamihira has not been collected here. Some of these writers are much more explicit and have spoken of even multiplication and division by zero. Among the earlier writings also there are two hymns of the Atharveda[6] where, most probably, there is reference to the zero as well as to positive and negative numbers. They will, however, form the subject of a future communication. For the present it may be stated that in the hymns referred to, the zero has been called *kṣudra* (trifling)[7] and positive and negative (numbers) have been denoted by *ṛca* and *anṛca* respectively.

My thanks are due to Professor David Eugene Smith of Columbia University for his interest in the preparation of this paper.

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[1] Ch. viii, sūtras 29, 30. Weber translates them as "Bei Eins eine Null" and "Zweimal bei einer Null." For his commentary of the sūtras see, *loc. cit.*, *pp.* 444-48.

[2] *Agni-Purāna*, Bangabāshī edition, Ch. 328–34.

[3] "Was denn auch in der That wohl keinem Zweifel unterliegt" (*Br.*p. (445). *Cf.* also p. 169.

[4] Winternitz, *Geschichte der Indischen Litteratur*, Bd. iii (1922) p. 28; compare also Max Müller, *Ancient Sanskrit Literature*, p. 75 et seq.

[5] This will be at once evident from its being raised to the level of a *Vedāṅga*, which mark it does not in reality deserve, from its being incorporated in the *Agni-Purāna* as already stated, and also from its being quoted in the *Bhāratīya-Nātya-Sūtra* (Ch. xv) and in the *Pariśiṣṭhas*. Vide Max Müller, *loc cit.*

[6] xix. 22, 23.

[7] *Cf.* Amarsiṅha's synonym for zero as *tucca*, meaning insignificant, negligible. Perhaps a better or more succinct definition is hardly possible even today. Amarsiṅha is the celebrated lexicographer of ndia of the fifth century of the Christian era.

NUMERICAL INTEGRATION OF ORDINARY DIFFERENTIAL EQUATIONS

By W. E. MILNE, University of Oregon.

The method of numerical integration here described has grown out of the practical experience of the writer in the course of a long series of integrations carried on for several years. In its fundamental principles this method does not differ essentially from other well-known methods,[1] but in practical operation it differs considerably. A careful trial of the method in comparison with others convinces me that it possesses distinct advantages in ease, speed, and simplicity. It is especially designed so that every step (except perhaps the substitution in the differential equation) may be readily performed on a calculating machine, thus preventing mental fatigue and insuring accuracy.

1. Let u_0, u_1, u_2, u_3, u_4, be five values of a function $u(x)$ at equally spaced values of x, which we may without loss of generality take to be $0, h, 2h, 3h, 4h$. A polynomial of fourth degree which takes the values u_0, u_1, u_2, u_3, u_4, is given by Newton's interpolation formula

$$P(x) = u_0 + \Delta u_0 \frac{x}{h} + \Delta^2 u_0 \frac{x(x-h)}{2h^2} + \Delta^3 u_0 \frac{x(x-h)(x-2h)}{6h^3} + \Delta^4 u_0 \frac{x(x-h)(x-2h)(x-3h)}{24h^4}. \quad (1)$$

From this we obtain by integration and reduction

$$\int_{x_0}^{x_4} P(x)dx = \frac{h}{3}\{u_4 + 4u_3 + u_2\} - 8h\Delta^4 u_0/720, \quad (2)$$

$$\int_{x_0}^{x_4} P(x)dx = \frac{4h}{3}\{2u_3 - u_2 + 2u_1\} + 224h\Delta^4 u_0/720. \quad (3)$$

Since

$$\Delta^4 u = h^4 \frac{d^4 u}{dx^4} + \textit{terms of higher order}$$

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[1] The following references may be consulted in this connection:

(1) Von Sanden, *Practical Mathematical Analysis*, E. P. Dutton and Co., Chapters X and XI.

(2) Whittaker and Robinson, *The Calculus of Observations*, London, 1924, Chapter XIV.

(3) Runge und König, *Numerisches Rechnen*, Berlin, 1924, Kapitel X.

(4) *The Method of Numerical Integration in Exterior Ballistics*, Ordnance Textbook, Washington, D. C., 1919.

if $u(x)$ is analytic for the interval under consideration it will be possible to take h small enough that the last terms in (2) and (3) can be neglected. We shall then write in general

$$\int_{x_{n-2}}^{x_n} u(x)dx = \frac{h}{3}\{u_n + 4u_{n-1} + u_{n-2}\}, \tag{4}$$

$$\int_{x_{n-4}}^{x_n} u(x)dx = \frac{4h}{3}\{2u_{n-1} - u_{n-2} + 2u_{n-3}\}. \tag{5}$$

Formula (4) is Simpson's rule. Formula (5) is less accurate, as is seen from the remainder terms in (2) and (3) but for our purposes possesses the advantage that the integral can be calculated without the knowledge of u_n.

2. Suppose that we are to solve a differential equation

$$\frac{dy}{dx} = f(x, y) \tag{6}$$

with the initial value $y=y_0$ when $x=x_0$. Suppose also that a suitable interval h has been chosen, and that the values of y have been obtained for $x_1=x_0+h$, $x_2=x_0+2h$, and $x_3=x_0+3h$. As the calculation of these initial values involves special methods we shall defer consideration of them to the next paragraph. The calculated values may be written as in the margin. To proceed with the computation we obtain a trial value of y_4 by use of the formula (5), which in this notation becomes

x	y	y'
x_0	y_0	y_0'
x_1	y_1	y_1'
x_2	y_2	y_2'
x_3	y_3	y_3'
x_4		

$$y_4 = y_0 + \frac{4h}{3}\{2y_3' - y_2' + 2y_1'\}. \tag{7}$$

We then obtain a trial value of y_4' by substitution in (6). Now we recalculate y_4 using formula (4), which becomes

$$y_4 = y_2 + \frac{h}{3}\{y_4' + 4y_3' + y_2'\}. \tag{8}$$

If the values given by (7) and (8) check to the desired number of decimal places we take this value as correct and proceed to the calculation of y_5 in a similar manner. If the values given by (7) and (8) do not check then either an error has been committed in calculation or the fourth differences of y' are so large that the remainder terms in (2) and (3) affect the results. Since the total discrepancy ϵ due to this cause is $\epsilon=232h\Delta^4y'/720$, the error ϵ_1 of formula (4) due to the fourth difference is

$$\epsilon_1 = \epsilon/29. \tag{9}$$

Consequently if the values given by (7) and (8) have a difference ϵ we first make sure that this is not due to an error of calculation, and if not, we calculate ϵ_1 by (9). If ϵ_1 is negligible to the number of decimal places desired we assume that the value of y_4 given by (8) is correct and proceed to the calculation of y_5 in a similar manner.

Of course it is necessary to obtain the corrected value of y_4' by direct calculation from the differential equation (6). The change in y_4' due to a change ϵ in y_4 is approximately $\epsilon\,\partial f/\partial y$, and the change ϵ_2 in y_4 due to this correction is therefore approximately $\epsilon_2 = (\epsilon h/3)(\partial f/\partial y.)$ If ϵ_2 is large enough to be significant the value of y_4 should be recomputed by (4) after y_4' has been corrected, but in many cases it is found that this step is unnecessary.

When the value of ϵ_1 is significant we must assume that $\Delta^4 y'$ is too large for the remainder terms in (2) and (3) to be ignored, and therefore we must take a smaller value for the interval h.

In actual practice these checks and corrections which it takes some time to describe can be performed very readily and (except when actual mistakes have been committed) take practically no additional time. It is desirable to record the values of the discrepancies ϵ in a column beside the values of y. Any suspicious fluctuation in the successive values of ϵ is an indication of a mistake in calculation, and then, and only then, we recheck our work far enough back to discover and remove the cause of the fluctuation.

As a very simple example let us obtain a few values of the integral of

$$\frac{dy}{dx} = 1.2y$$

with the condition that $y=1$ when $x=0$. The method of getting the first four values is explained later. All values after the first four are obtained by the process just explained.

x	y	ϵ	y'
−0 1	.88692		1.06430
.0	1.00000		1.20000
.1	1.12750		1.35300
.2	1.27125		1.52550
.3	1.43333	.00001	1.72000
.4	1.61607	1	1.93928
.5	1.82212	1	2.18654
.6	2.05443	1	2.46532
.7	2.31637	2	2.77964
.8	2.61169	1	3.13403
.9	2.94468	2	3.53362

The given differential equation has the solution

$$y_1 = e^{1.2x}$$

values of which may be obtained from existing tables. It is found that the values given by the method of numerical integration check in a satisfactory manner.

3. Various ways may be suggested for finding the first four values of the computation. One method which has proved very successful in practice is to calculate from the differential equation (6) the initial values y_0', y_0'', etc., from which we get the Taylor's series

$$y = y_0 + y_0'(x - x_0) + \frac{y_0''}{2!}(x - x_0)^2 + \cdots . \tag{10}$$

If as often happens, this series converges rapidly for the first two or three values of x, the corresponding values of y can be easily obtained. This method is available only when the solution is analytic at $x = x_0$.

A second method which often is very satisfactory is a method of successive approximations. For it, we require another integral formula, obtained from (1) by integration from x_2 to x_3. It is

$$\int_{x_2}^{x_3} P(x)dx = \frac{h}{24}\{-u_4 + 13u_3 + 13u_2 - u_1\} + \frac{11h\Delta^4 u_0}{720}. \tag{11}$$

We may assume that the remainder term is negligible, and with a slight change of subscripts write (11) in the form

$$\int_{x_0}^{x_1} u(x)dx = \frac{h}{24}\{-u_2 + 13u_1 + 13u_0 - u_{-1}\}. \tag{12}$$

To get started on the computation we assume trial values for y_{-1}, y_1, y_2 as follows:

$$y_{-1} = y_0 - hy_0', \quad y_1 = y_0 + hy_0', \quad y_2 = y_0 + 2hy_0',$$

and then calculate from (6) the corresponding trial values of y_{-1}', y_1', y_2'. From these we recompute y_2, y_1, y_{-1} by the formulas

$$\begin{aligned} y_2 &= y_0 + \frac{h}{3}\{y_2' + 4y_1' + y_0'\}, \\ y_1 &= y_0 + \frac{h}{24}\{-y_2' + 13y_1' + 13y_0' - y_{-1}'\}, \\ y_{-1} &= y_1 - \frac{h}{3}\{y_1' + 4y_0' + y_{-1}'\}. \end{aligned} \tag{13}$$

We again recompute y_{-1}', y_1', y_2' by (6) and y_{-1}, y_1, y_2 by (13) and continue the process until the values are unchanged by further recomputation.

For example in the problem of paragraph 2 we have

First approximation	−.1	.88000	1.05600
	0	1.00000	1.20000
	.1	1.12000	1.34400
	.2	1.24000	1.48800
Second approximation	−.1	.88720	1.06464
	0	1.00000	1.20000
	.1	1.12720	1.35264
	.2	1.26880	1.52256
Third approximation	−.1	.88691	1.06429
	0	1.00000	1.20000
	.1	1.12749	1.35299
	.2	1.27110	1.52532
Fourth approximation	−.1	.88692	1.06430
	0	1.00000	1.20000
	.1	1.12750	1.35300
	.2	1.27124	1.52549
Fifth approximation	−.1	.88692	1.06430
	0	1.00000	1.20000
	.1	1.12750	1.35300
	.2	1.27125	1.52550

The figures which are changed by recomputation are indicated by bold faced type.

4. This completes the discussion of the process. A few remarks may however be useful.

First, in actual practice we should make use of an additional check, the theory of which is as follows. Since Simpson's rule applies to an even number of intervals, we easily see that the values of y corresponding to even values of x are obtained by addition to preceding even values, while those corresponding to odd values are obtained by addition to preceding odd values. Hence it may come about that the accumulation of errors too small to be detected by the values of ϵ may produce an oscillation in the values of y. This may be checked by the use of an integral formula for an odd number of intervals. One of the simplest formulas is obtained from (1) by integration from x_1 to x_4 which gives

$$\int_{x_1}^{x_4} P(x)dx = \frac{h}{24}\{8u_4 + 31u_3 + 21u_2 + 13u_1 - u_0\} + \frac{3h}{720}\Delta^4 u_0. \quad (14)$$

If this formula is applied to every third line of the computation it is almost impossible for errors to escape detection.

Second, the division by 3 which must be performed in (7) and (8), except when h is divisible by 3, is done on the computing machine as follows. The dividend is on the dials of the machine as a result of the previous operation. The operator then divides by 3 mentally, recording the result on the keyboard, and then substracts three times, which affords a sure check. The quotient

is now on the keyboard ready for addition to (or subtraction from) the proper value of y.

Third, the proper value of h to secure a given degree of accuracy can be calculated theoretically by appropriate formulas of finite differences before the computation is begun, but in practice it is usually just as satisfactory to arrive at the best value by trial and error. Moreover it is often desirable to change the size of h in the course of a computation, because of a considerable change in the fourth differences.

Fourth, the method applies equally well to solutions of differential equations of the second order, and was in fact devised especially for a second order equation. The steps are as follows. We have given a differential equation

$$y'' = f(x, y, y') \tag{15}$$

and obtain the first four values by series or by successive approximations. Then we get a trial value of y' by the formula

$$y_4' = y_0' + \frac{4h}{3}\{2y_3' - y_2' + 2y_1''\}.$$

Then using this value of y_4' we compute a trial value of y_4 by Simpson's rule, then calculate y_4'' by (15), then recalculate y_4' by Simpson's rule, and continue until a satisfactory check is reached.

QUESTIONS AND DISCUSSIONS

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The Department of Questions and Discussions in the Monthly is open to all forms of activity in collegiate mathematics, including the teaching of mathematics, except for specific problems, especially new problems, which are reserved for the separate department of Problems and Solutions

DISCUSSIONS

I. NEW PROPERTIES OF AN ORTHOCENTRIC SYSTEM OF TRIANGLES.

By A. A. BENNETT, Lehigh University.

Let T_1, T_2, T_3, be the vertices of an arbitrary triangle in a given plane. We shall introduce certain points in this plane associated with the triangle as follows, (where in each case $i = 1, 2, 3$),

M_i, the mid-point of the side $T_j T_k$, $i \neq j \neq k$,

T_0, the orthocenter of the triangle T_1, T_2, T_3,

M_i', the mid-point of $T_0 T_i$,

P_i, the foot of the perpendicular from T_i on $T_j T_k$,

N, the mid-point of $M_1 M_1'$, (let $N M_1 = n$).

T_0', T_1', T_2', T_3', P_1', P_2', P_3', the reflexions in N of the respective points in unprimed letters.

This system of points is called an *orthocentric system.*

The following relations are well-known and may be found for example in Altshiller-Court's *College Geometry.*

1. The lines M_1M_1', M_2M_2', M_3M_3', meet and bisect each other.

2. If one starts with the triangle $T_1'\ T_2'\ T_3'$, instead of with the given triangle $T_1T_2T_3$, the same set of six points, M_1, M_1', M_2, M_2', M_3, M_3', would have been obtained but with the primed and unprimed letters exchanging rôles.

3. Each of the four points, T_0, T_1, T_2, T_3, is the orthocenter of the triangle of the other three. Similarly for T_0', T_1', T_2', T_3'.

4. T_h, $(h=0, 1, 2, 3)$, is the center of a circle of fixed common radius, $2n$, passing through T_i', T_j', T_k', h, i, j, k, all distinct. Similarly for the same n as originally defined, for the primed and unprimed letters interchanged.

5. N is the center of a circle, the *nine-point circle,* of radius n, through the twelve points, P_i, P_i', M_i, M_i'.

6. In each of the twelve points, P_i, P_i', M_i, M_i', meet two mutually perpendicular sides, one from the quadrilateral, $T_0\ T_1\ T_2\ T_3$ and one from the quadrilateral, $T_0'\ T_1'\ T_2'\ T_3'$.

It is easy to verify also the following,

7. The vector NT_0' is the sum of the vectors NT_1, NT_2, NT_3. Also in every case the vector from N to a vertex, is the sum of the vectors from N to the three medial points, M, or M', of the sides meeting in this vertex.

8. The point P_1 is the reflexion of M_1 in a line through N for which M_2 and M_3 are mutual reflexions. Similarly for the other points, P_2 and P_3.

Some new relations verified by computations with concomitants of the simultaneous cubic and quadric binary forms are the following:

There is a one-parameter linear family of cubic curves, K, each symmetric with respect to N, and each passing through the nine pointss, N, T_0, T_1, T_2, T_3, T_0', T_1', T_2', T_3'. One of these, K_M, meets the nine point circle, C, in the six medial points, M_i, M_i'. Another, K_p, meets C in the six pedal points, P_i, P_i'. A third, K_c, is a circular cubic. Furthermore the real asymptote, L, of K_c, is also the inflexional tangent at N to K_M, and is also the second polar with respect to the two circular points at infinity of the "medial line triple," NM_1, NM_2, NM_3.

There is a host of remarkable points and lines and also several classical cubic curves associated with a triangle. Most of these are so directly associated with the triangle, that they cannot be regarded as determined symmetrically for the entire orthocentric system. In contrast to these celebrated related figures it is of special interest to point out that the line L and the cubics, K_M,

K_p, and K_c, are common property of the related eight triangles, $T_0T_1T_2$, $T_1T_2T_3$, $T_2T_3T_0$, $T_3T_0T_1$, $T_0'T_1'T_2'$, $T_1'T_2'T_3'$, $T_2'T_3'T_0'$, $T_3'T_0'T_1'$, and are determined in the same position by any one of these eight triangles alone.

Since the reader may desire to check these results or continue the investigation along these same lines, we shall here list the more important relations in explicit form. To do this in an economical manner, we take the medial line triple and nine point circle as fundamental. It is of significance that here the computation deals essentially with binary forms, rather than with the ternary forms which are required for the study of the other special figures usually studied in connection with the triangle.

1. Nine point circle, $C=0$, $C\equiv x^2+y^2-1$,
2. Medial line triple, $F=0$, $F\equiv ax^3+3bx^2y+3cxy^2+dx^3$,
3. The line, $L=0$, $L\equiv(a+c)\,x+(b+d)y$,
4. The cubic, $K_M=0$, $K_M\equiv 4F-3CL$,
5. The pedal line triple, NP_1, NP_2, NP_3, written in product form $P=0$,

$$\begin{aligned}P\equiv\ &(8ad^2-12abd-36bcd-a^3+9ac^2-3a^2c+27c^3)x^3\\&+(9a^2d-12bd^2+9c^2d-18abc-18acd-27bc^2-3a^2b+36b^2d)x^2y\\&+(9ad^2-12a^2c+9ab^2-18bcd-18abd-27b^2c-3cd^2+36ac^2)xy^2\\&+(8a^2d-12acd-36abc-d^3+9b^2d-3bd^2+27b^3)y^3,\end{aligned}$$

6. A second fundamental associated line, $L'=0$,

$$\begin{aligned}L'\equiv\ &(ad^2+ab^2-4bcd-a^2c+3c^3+2ac^2-2abd)x\\&+(a^2d+c^2d-4abc-bd^2+3b^3+2b^2d-2acd)y,\end{aligned}$$

7. The cubic, K_p, $K_p=0$, where K_p can be written in either of the two forms, $P+9CL'$, and $-[(a-3c)^2+(d-3b)^2]F+9L'$.

II. On a Circle Attached to a Collapsible Four-Bar

By Helen Barton, Johns Hopkins University.

1. If in a quadrangle of four jointed rods, three of them are made to move in a plane about the fourth, which remains fixed, the resulting configuration is known as the general case of "Three Bar Motion".[1] If, however, we impose the condition that $p_1\pm p_2\pm p_3\pm p_4=0$, where p_1 is the length of the first rod, etc., we have the rational case. When there is to be any motion of the rods, two of the terms in the above relation will be positive and two negative. It is with this rational case that the problem here presented is connected.

[1] Morley, F. V. "An analytical Treatment of the Three-Bar Curve". *Proc. London Math. Soc.* Vol. 21. 1922. p. 140.

It is a known fact that if the four jointed rods are of such lengths that all are tangent to the same circle, then as they vary their relative positions, the rods will always be tangent to some circle. It is the purpose of this paper to determine the locus of the center of this circle and also to note certain characteristic properties of this collapsible quadrangle or four-bar.

Two cases naturally arise: (1) the circle is inscribed in the four-bar; (2) the circle is escribed to the four-bar.

2. In the first case, we know from the properties of the circle that the sum of the opposite sides of the circumscribed quadrangle must equal the sum of the other two sides. Thus, if we assign the numbers 1, 2, 3, 4 in order to the sides of the quadrangle, we have

$$p_1 + p_3 = p_2 + p_4. \tag{1}$$

These four jointed rods form a closed sequence or quadrangle, which may be regarded as a sequence of points and strokes. Thus if p_n = the length of any side, and θ_n = the angle between that directed side and the base line, then since the sequence of lines is closed, we have

$$\sum_{n=1}^{4} p_n e^{i\theta_n} = \sum_{n=1}^{4} p_n t_n = 0, \text{ where } t_n = e^{i\theta_n}.$$

If now we choose p_1 as the stationary rod and also as a segment of our base line, our equation becomes

$$p_1 + p_2 t_2 + p_3 t_3 + p_4 t_4 = 0. \tag{2}$$

This equation carries with it its conjugate equation, that is, the equation derived by replacing every term in (2) by its conjugate or image in the base line. Hence

$$p_1 + p_2 t_2^{-1} + p_3 t_3^{-1} + p_4 t_4^{-1} = 0. \tag{3}$$

To find the center of the inscribed circle, we must bisect the two interior angles at either end of p_1, and since this does not involve the side p_3, we shall eliminate t_3 and p_3 from equations (2) and (3). Eliminating t_3, we obtain

$$p_1^2 + p_2^2 + p_4^2 + p_1p_2(t_2 + t_2^{-1}) + p_1p_4(t_4 + t_4^{-1}) + p_2p_4(t_2t_4^{-1} + t_4t_2^{-1}) = p_3^2. \tag{4}$$

If we represent the interior angle between p_1 and p_4 by 2α and the interior angle between p_2 and p_1 by 2β we have

$$t_2 = e^{i(\pi - 2\beta)} = \cos(\pi - 2\beta) + i \sin(\pi - 2\beta),$$

$$t_2^{-1} = e^{-i(\pi - 2\beta)} = \cos(\pi - 2\beta) - i \sin(\pi - 2\beta).$$

Hence $$(t_2 + t_2^{-1}) = 2\cos(\pi - 2\beta) = -2\cos 2\beta,$$

Similarly $$(t_4 + t_4^{-1}) = 2\cos(\pi + 2\alpha) = -2\cos 2\alpha.$$
$$(t_2 t_4^{-1} + t_4 t_2^{-1}) = 2\cos(2\alpha + 2\beta).$$

Making these substitutions in (4), we obtain

$$p_1^2 + p_2^2 + p_4^2 - 2p_1p_2\cos 2\beta - 2p_1p_4\cos 2\alpha + 2p_2p_4\cos(2\alpha + 2\beta) = p_3^2.$$

Squaring (1) we have

$$p_1^2 + p_2^2 + p_4^2 - 2p_1p_2 - 2p_1p_4 + 2p_2p_4 = p_3^2.$$

Hence $$p_1p_2(1 - \cos 2\beta) + p_1p_4(1 - \cos 2\alpha) - p_2p_4[1 - \cos(2\alpha + 2\beta)] = 0,$$

or, $$p_1p_2\sin^2\beta + p_1p_4\sin^2\alpha - p_2p_4\sin^2(2\alpha + 2\beta) = 0.$$

It follows then from the law of sines that

$$\text{or} \qquad \begin{aligned} p_1p_2r_1^2 + p_1p_4r_2^2 - p_1^2p_2p_4 &= 0, \\ p_2r_1^2 + p_4r_2^2 - p_1p_2p_4 &= 0, \end{aligned} \tag{5}$$

when r_1, r_2 are the distances of the center of the inscribed circle from the two extremities of the rod p_1; r_1 being the one associated with the angle 2α and r_2 with the angle 2β.

If now, we choose the intersection of p_1 and p_4 as the origin and p_1 as coincident with the x axis, we have the following relations between r_1, r_2 and x and y, the rectangular coordinates of the center of the circle:

$$r_1^2 = x^2 + y^2, \quad r_2^2 = (p_1 - x)^2 + y^2.$$

Hence equation (5) becomes

$$\begin{aligned} p_2(x^2 + y^2) + p_4[(p_1 - x)^2 + y^2)] &= p_1p_2p_4, \\ (x^2 + y^2)(p_2 + p_4) - 2p_1p_4x + p_1^2p_4 - p_1p_2p_4 &= 0. \end{aligned} \tag{6}$$

Thus we see that the locus of the center of the inscribed circle is a circle, whose center is $(p_1p_4/(p_2+p_4),0)$ and radius equals $\sqrt{p_1p_2p_3p_4}/(p_2+p_4)$.

3. In the second case, where the circle is escribed to the four-bar, the procedure is very similar to the one just considered, Here we have the following relation existing between the sides

$$p_1 + p_4 = p_2 + p_3. \tag{7}$$

Now if we keep p_1 fixed as before, and vary the relative positions of the other three rods, it is evident that the rods may form a convex quadrangle or may become crossed, but the analytical treatment following holds for each type.

We shall use p_1 as the base line as before, then equations (2), (3), (4) hold as in the first case. Our angles, however, are slightly different, for to obtain the center (x, y) of the escribed circle, we must bisect the interior angle at one extremity of p_1 and the exterior angle at the other end.

By steps similar to those which led to equation (5), we obtain in this case

$$p_2r_1^2 - p_4r_2^2 + p_1p_2p_4 = 0. \tag{8}$$

Transforming to Cartesian coordinates, with the intersection of p_1 and p_4 as origin, and p_1 coinciding with the x axis, we obtain as the equation of the locus of the center of the escribed circle

$$(x^2 + y^2)(p_2 - p_4) - 2x(p_1p_4) + p_1p_2p_4 - p_1^2p_4 = 0. \tag{9}$$

Thus the locus in this case is also a circle, with center $(p_1p_4/(p_2-p_4),0)$ and radius $\sqrt{p_1p_2p_3p_4}/(p_2-p_4)$.

4. In each of the two cases considered, we note that the centers of the two loci fall on the fixed rod and that the centers divide the rod p_1 internally and externally in the ratio of p_4 to p_2.

If now we had kept p_2 fixed and allowed the other rods to vary their positions, the locus would have been a circle about a point on p_2, with radius $\sqrt{p_1p_2p_3p_4}/(p_1+p_3) = \sqrt{p_1p_2p_3p_4}/(p_2+p_4)$. That is, the radius of the circle, which is the locus, is a constant, no matter which rod is kept fixed. Hence, if we divide each of the four rods internally in the ratio of adjacent rods, the four points so obtained are at a constant distance from the center of the inscribed circle. Hence, as the rods vary their positions these four points always lie on a circle. The same holds true for the points obtained by dividing the rods externally in the ratio of adjacent rods.

5. It has been shown that:

given four jointed bars of such lengths that they are all tangent to the same circle,

(1) as three of these bars move about the fourth, the locus of the center of the circle, whether inscribed or escribed, is a complete circle with center on the fixed bar;

(2) in the inscribed case, the center of the locus divides the fixed bar internally in the ratio of adjacent bars; in the escribed case, the center of the locus divides the fixed bar externally in the ratio of adjacent bars;

(3) the four points of tangency of the maximum inscribed circle (or escribed) always lie on a circle with constant radius

$$\frac{\sqrt{p_1p_2p_3p_4}}{p_2 + p_4}\left(\text{or } \frac{\sqrt{p_1p_2p_3p_4}}{p_2 - p_4}\right),$$

no matter what the relative positions of the rods may be.

III. A Problem of Regions

By H. A. Robinson, Johns Hopkins University

1. Questions as to the number of ovals of plane curves called attention to the following analysis. The results obtained will enable one to find the number of regions formed by a number of planes in space, or more generally to the number of regions formed by a number of m-dimensional flat space cuts in an $m+1$ dimensional space.[1]

If a line has no *cuts* on it, it may be said to consist of one 1-dimensional region. One point or 0-dimensional cut [0-cut] on a line divides the line into two regions, two cuts into three regions, and obviously n cuts (no two cuts coinciding) into $n+1$ regions.

A plane free of cuts may be said to form one 2-dimensional region. One line or 1-dimensional cut [1-cut] divides the plane into two regions; two cuts, four regions[2]; three cuts, seven regions—one of which is closed[3] and the other six open; four cuts, eleven regions—three of which are closed and eight open; etc. Now it may be readily observed that the added nth line or 1-cut, intersects the existing 1-cuts in $n-1$ points or 0-cuts. The n 1-dimensional regions formed by the $n-1$ 0-cuts on the nth 1-cut pass through n existing 2-dimensional regions, and hence add n new regions to the plane. Therefore the number of regions which n 1-cuts form in a plane satisfies the equation

$$I^1_n = I^1_{n-1} + I^0_{n-1}$$

where I represents the number of regions; the superscript, the dimension of the cutting space; and the subscript, the number of cuts. Since $I^0_{n-1} = n$, we have

$$I^1_n - I^1_{n-1} = n.$$

Solving this partial difference equation and adjusting the constant, we find

$$I^1_n = 1 + \tfrac{1}{2}\, n(n+1).$$

A three way space free of plane or 2-cuts may be said to form one 3-dimensional region. One 2-cut divides the space into two regions; two cuts, four regions; three cuts, eight regions; four cuts, fifteen regions—one of which is closed and fourteen open; five cuts, twenty-six regions—four of which are

[1] *Cf.* Steiner, "Über die Teilung der Ebene und des Raumes," *Crelle*, vol. 1, p. 349. A. J. Kempner.

[2] Through the whole article, it will be assumed that every cutting space will intersect every other cutting space the maximum number of times. Thus every line in the plane will intersect every other line in the plane. Also no three lines shall meet in a point, for if they did we would not have the maximum number of regions.

[3] By closed regions we mean regions which are entirely bounded by cutting spaces.

closed; etc. Now it may be noted that the added nth 2-cut, intersects the existing $n-1$ 2-cuts in $n-1$ 1-cuts. These $n-1$ 1-cuts form on the nth 2-cut I^1_{n-1} plane regions and hence add I^1_{n-1} new regions to the three way space. Therefore the number of regions formed by n 2-cuts in the space in question may be represented by the following equation

$$I^2_n = I^2_{n-1} + I^1_{n-1},$$

the solution of which is (constant adjusted),

$$I^2_n = \tfrac{1}{6}\, n(n+1)(n+2) - \tfrac{1}{2}\,(n-2)(n+1).$$

There is no logical reason why the investigation should stop here. An $m+1$ way space free of m-cuts may be thought of as forming one $m+1$ dimensional region. One m-cut divides the space into two regions; two cuts, four regions; etc. The first $m+1$ cuts form no closed regions; the next cut would form one closed region; the succeeding, $m+2$ closed regions; etc. The added nth m-cut intersects the existing $n-1$ m-cuts in $n-1$ $(m-1)$-cuts. These $n-1$ $(m-1)$-cuts form on the nth m-cut I^{m-1}_{n-1} regions and hence add I^{m-1}_{n-1} new regions to the $m+1$ way space. Therefore the number of regions formed by n m-cuts in an $m+1$ way space may be represented by the following equation

$$I^m_n = I^m_{n-1} + I^{m-1}_{n-1}.$$

Let us tabulate the results found. Table A gives the number of regions produced by a given number of cuts; Table B, the number of open regions

TABLE A

The Total Number of Regions Formed by I cuts

n	I^0_n	I^1_n	I^2_n	I^3_n	I^4_n	.....
0	1	1	1	1	1	
1	2	2	2	2	2	
2	3	4	4	4	4	
3	4	7	8	8	8	
4	5	11	15	16	16	
5	6	16	26	31	32	
6	7	22	42	57	63	
...	...	....	....	....	....	.....

TABLE B

The Number of Open Regions Formed by I cuts

n	oI^0_n	oI^1_n	oI^2_n	oI^3_n	oI^4_n	.....
0	1	1	1	1	1	
1	2	2	2	2	2	
2	2	4	4	4	4	
3	2	6	8	8	8	
4	2	8	14	16	16	
5	2	10	22	30	32	
6	2	12	32	52	62	
...	...	....	....	....	....	.....

TABLE C

The Number of Closed Regions Formed by I cuts

n	cI^0_n	cI^1_n	cI^2_n	cI^3_n	cI^4_n	.....
0	0	0	0	0	0	
1	0	0	0	0	0	
2	1	0	0	0	0	
3	2	1	0	0	0	
4	3	3	1	0	0	
5	4	6	4	1	0	
6	5	10	10	5	1	
...	...	....	....	....	....	.....

formed; and Table C, the number of closed regions. Upon examination of Table C we note that its matrix contains the numbers of Pascal's triangle, *i.e.*, the binomial coefficients (with the exception of a border of ones on the left hand side). Each element of Table B is a linear combination of those found in C. Now the general law, as may be read from the tables, for the total number of regions is

$$I_n^m = \sum 2^k C_{n-k-1}^{m-k+1},$$

where C_s^r means the coefficient found in the sth row and the rth column of *Pascal's matrix* (*i.e.*, Pascal's triangle made into rectangle form by the addition of zeros), and k runs from 0 to the particular value that will make first the superscript or subscript zero. In order to illustrate the notation, let us consider the following example. Given six planes in a three way space, of which every plane intersects every other plane.[1] The number of regions formed is

$$I_6^3 = 2^0C_5^3 + 2^1C_4^2 + 2^2C_3^1 + 2^3C_2^0$$

$$= 1\cdot10 + 2\cdot6 + 4\cdot3 + 8\cdot1 = 42.$$

2. The above has dealt with a system of cutting spaces that intersect each other in *one* point, *one* line, etc. We have denoted the number of regions by I. Let us now consider circles (and their logical extension to higher spaces) as our cutting spaces in a plane and let us denote the number of regions formed in the plane by II_n^1. Since two circles intersect in at most *two* real points, we will here denote the number of regions by II.

A circle free of *cuts* may be said to consist of one arc or region. One *two point* or (0)-cut on a circle divides the circle into two regions; two cuts, four regions; three cuts, six regions; etc. Obviously n cuts form $2n$ regions.

A plane free of circle-cuts or (1)-cuts may be said to consist of one two-dimensional region. One (1)-cut divides the plane into two regions; two cuts, four regions; three cuts, eight regions; four cuts, fourteen regions; etc. We note now that the existing $n-1$ (1)-cuts intersect the nth (1)-cut in $n-1$ (0)-cuts. These $n-1$ (0)-cuts on the circle form II_{n-1}^0 arcs and hence add II_{n-1}^0 new regions to the plane. Therefore

$$II_n^1 = II_{n-1}^1 + II_{n-1}^0,$$

the solution of which is

$$II_n^1 = n(n-1) + 2.$$

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[1] This problem is given in many puzzle books: How many pieces of cheese will six straight cuts of a knife divide a cake into?

When circles are *opened out* into lines the number of regions lost is equal to the number of closed regions found in the case of lines in a plane. Hence we have a way to pass from line to circle regions.

In a three way space, spheres [(2)-cuts] would be used as the cutting spaces.

The extension may be made to an $m+1$ way space. The number of regions formed by n (m)-cuts would be given by

$$II_n^m = II_{n-1}^m + II_{n-1}^{m-1}.$$

When the (m)-cuts are *opened out* into m-cuts the number of regions lost is equal to the number of closed regions found in case of m-cuts. Hence

$$II_n^m = I_n^m + \text{Closed } I_n^m = [\sum 2^k C_{n-k-1}^{m-k+1}] + C_{n-1}^{m+1}.$$

Table D gives the number of regions produced by a given number of (i)-cuts in an $i+1$ way space.

TABLE D

The Total Number of Regions Formed by II cuts

n	II_n^0	II_n^1	II_n^2	II_n^3	II_n^4	.....
0	1	1	1	1	1	
1	2	2	2	2	2	
2	4	4	4	4	4	
3	6	8	8	8	8	
4	8	14	16	16	16	
5	10	22	30	32	32	
6	12	32	52	62	64	
...	...	....	....	....	....	.....

It would be interesting to have an extension of this notion to A_n^m, where $A > II$. [A is the number of times an initial cutting space would intersect another initial cutting space.]

IV. A MODIFIED METHOD FOR CUBE ROOTS AND FIFTH ROOTS

By L. S. DEDERICK, St. Stephen's College

In a recent number of this MONTHLY (*1925*, 377) Mr. D. H. Lehmer gave a short method of extracting cube roots and fifth roots with the help of the calculating machine, the essential feature being to express the number as a product of factors of the form $1+p.10^{-n}$ where p is an integer less than 10. The purpose of this note is to present a method which is similar but even shorter. The difference is that the factors are the *reciprocals*[1] of a similar set

[1] It seems probable that this is the method intended in the article on logarithms in the *Encyc. Brit.* to which Mr. Lehmer gives a reference. There is in that article an obvious error, which Mr. Lehmer apparently aims to correct by a change of sign. It seems more probable, however, that the correction should be a transfer of the factor to the denominator.

of simple numbers, given in the accompanying table, and hence may be taken out by *multiplication* instead of division. A rule for the 'factoring' may be formulated thus:

Multiply or divide the number by such a power of 10 *that the result shall lie between 1 and 10. Multiply the result by the smallest number n in the table that will leave it greater than 1. Repeat this process until the result is sensibly unity.*

The factors then are a power of 10 and the reciprocals of the numbers by which we have multiplied. The cube roots and fifth roots of these reciprocals are accordingly the numbers tabulated, and the factors of the required root are such of these as correspond to the multipliers used.

The shortening of the work is greater than merely the substitution of multiplication for division. The first multiplication will in any case produce a result less than 1.2. The multiplications after that will be by numbers greater than or equal to 0.9. Each of these multiplications may be performed by *subtracting* a multiple of the number in the machine by a number of one digit. Thus to multiply by .94 we subtract .06 of the number. The 6 will appear in red on the counting dials and should be left there, since the next multiplier is regularly in the next decimal place. This means that the counting dials need be cleared (if at all) only after the first multiplication. After that a succession of red digits on these dials such as 13609 means $[.9\times.97\times.994\times.99991]^{-1}$. For convenience of reference, the latter part of the table is made to agree with this, the entry in the first column being $1-n$, the same digit which appears on the counting dials. Of course after each multiplication the new product is transferred to the keys.

As in the other method, after half of the factors are determined, the remaining half can be found by inspection. It makes the work shorter, however, as well as more accurate, to observe that at this stage we may substitute $1+\frac{1}{3}x$ for $\sqrt[3]{1+x}$ and $1+\frac{1}{5}x$ for $\sqrt[5]{1+x}$. This enables us to replace this second group of factors by a single one of which the root may be written without even consulting the table.

These simplifications have made it appear worth while to compute the accompanying table to 16 significant figures. This provides for work involving two complete periods on an eight-column machine or a period and a half on one with ten columns. The latter case is mentioned because, if the number of figures does not exceed the number of columns of keys by more than one half, the work of multiplying is much shorter than in the contrary case.

The following examples are on an eight-column machine.

Example 1. To find the cube root of 165.97234. Multiplying this by .01 and 0.7 gives 1.1618064. Multiplying by 0.9, 0.96, etc. causes the red digits 14378 to appear on the counting dials, the next ones being 875. From the table we get the corresponding factors of the cube root as $4.6415888\times1.1262479\times1.0357442\times1.0137003\times1.0010020\times1.0002334$; and a final factor 1.0000296 is obtained by dividing .00008875 by 3. The product of these seven factors is 5.4955593, the required cube root.

Example 2. To find the fifth root of 4.8278165. The first multiplier is .23 and reduces the number to 1.1103978. For this 0.9 is too small and .91 (appearing as .09) gives 1.0104620. This illustrates the very unusual case where the next multiplier .99 appears in the same decimal place, namely as .01 on the dials. The mechanical effect is merely to convert the red 9 into a *black* 9. The rest of the factors present nothing unusual. In taking the factors from the table we merely need to remember that a black 9 signifies two factors belonging to the same decimal place, one with the digit 9, the other with the digit 1. The factors here then are $1.3416970 \times 1.0190411 \times 1.0020121 \times 1.0000715$ giving the root 1.3700934. If this is tested by raising to the fifth power, a correction is found necessary, the corrected value being 1.3700933.

RECIPROCALS OF CUBE AND FIFTH ROOTS

n	$1 \div \sqrt[3]{n}$	$1 \div \sqrt[5]{n}$	$1-n$	$1 \div \sqrt[3]{n}$	$1 \div \sqrt[5]{n}$
.0001		6.3095734 44801932	.0^{3}9	1.0^{3}3001 80126095	1.0^{3}1800 97264198
.001		3.9810717 05534973	.0^{3}8	1.0^{3}2668 08977442	1.0^{3}1600 76845085
.01	4.6415888 33612779	2.5118864 31509580	.0^{3}7	1.0^{3}2334 42281541	1.0^{3}1400 58830201
.1	2.1544346 90031884	1.5848931 92461113	.0^{3}6	1.0^{3}2000 80037352	1.0^{3}1200 43219017
.12	2.0274006 65191133	1.5281421 35815799	.0^{3}5	1.0^{3}1667 22243836	1.0^{3}1000 30011004
.14	1.9258567 85554179	1.4817481 47204298	.0^{3}4	1.0^{3}1333 68899954	1.0^{3}0800 19205634
.16	1.8420157 49320193	1.4426999 05907214	.0^{3}3	1.0^{3}1000 20004668	1.0^{3}0600 10802377
.18	1.7710976 15304352	1.4091119 53304333	.0^{3}2	1.0^{3}0666 75556939	1.0^{3}0400 04800704
.2	1.7099759 46676697	1.3797296 61461215	.0^{3}1	1.0^{3}0333 35555728	1.0^{3}0200 01200088
.23	1.6321399 23033617	1.3416969 50852690			
.26	1.5667831 20190969	1.3091979 91621230	.0^{4}9	1.0^{4}300 01800126	1.0^{4}180 00972064
.3	1.4938015 82185722	1.2722596 36539392	.0^{4}8	1.0^{4}266 68088977	1.0^{4}160 00768045
.35	1.4189834 11970384	1.2336341 72516721	.0^{4}7	1.0^{4}233 34422282	1.0^{4}140 00588030
.4	1.3572088 08297453	1.2011244 33981431	.0^{4}6	1.0^{4}200 00800037	1.0^{4}120 00432019
.45	1.3049558 80389621	1.1731606 76311841	.0^{4}5	1.0^{4}166 67222244	1.0^{4}100 00300011
.5	1.2599210 49894873	1.1486983 54997035	.0^{4}4	1.0^{4}133 33688900	1.0^{4}080 00192006
.6	1.1856311 01496688	1.1075663 43248290	.0^{4}3	1.0^{4}100 00200005	1.0^{4}060 00108002
.7	1.1262478 80443606	1.0739409 23785779	.0^{4}2	1.0^{4}066 66755557	1.0^{4}040 00048001
.8	1.0772173 45015942	1.0456395 52591273	.0^{4}1	1.0^{4}033 33355556	1.0^{4}020 00012000
.9	1.0357441 68651286	1.0212956 87600135			
			.0^{5}9	1.0^{5}30 00018000	1.0^{5}18 00009720
$1-n$			.0^{5}8	1.0^{5}26 66680889	1.0^{5}16 00007680
.09	1.0319362 51301859	1.0190411 49735922	.0^{5}7	1.0^{5}23 33344222	1.0^{5}14 00005880
.08	1.0281837 22701926	1.0168161 47821955	.0^{5}6	1.0^{5}20 00008000	1.0^{5}12 00004320
.07	1.0244851 88140280	1.0146199 80122523	.0^{5}5	1.0^{5}16 66672222	1.0^{5}10 00003000
.06	1.0208393 02540953	1.0124519 68893663	.0^{5}4	1.0^{5}13 33336889	1.0^{5}08 00001920
.05	1.0172447 68191101	1.0103114 59317936	.0^{5}3	1.0^{5}10 00002000	1.0^{5}06 00001080
.04	1.0137003 32595567	1.0081978 18497167	.0^{5}2	1.0^{5}06 66667556	1.0^{5}04 00000480
.03	1.0102047 86449813	1.0061104 34499387	.0^{5}1	1.0^{5}03 33333556	1.0^{5}02 00000120
.02	1.0067569 61723556	1.0040487 15456574			
.01	1.0033557 29847986	1.0020120 88709965	.0^{6}9	1.0^{6}3 00000180	1.0^{6}1 80000097
			.0^{6}8	1.0^{6}2 66666809	1.0^{6}1 60000077
.0^{3}9	1.0030181 26952430	1.0018097 84617414	.0^{6}7	1.0^{6}2 33333442	1.0^{6}1 40000059
.0^{3}8	1.0026809 77976795	1.0016077 25346310	.0^{6}6	1.0^{6}2 00000080	1.0^{6}1 20000043
.0^{3}7	1.0023442 81854107	1.0014059 10354030	.0^{6}5	1.0^{6}1 66666722	1.0^{6}1 00000030
.0^{3}6	1.0020080 37520976	1.0012043 39099701	.0^{6}4	1.0^{6}1 33333369	1.0^{6}0 80000019
.0^{3}5	1.0016722 43917573	1.0010030 11044186	.0^{6}3	1.0^{6}1 00000020	1.0^{6}0 60000011
.0^{3}4	1.0013368 99987618	1.0008019 25650083	.0^{6}2	1.0^{6}0 66666676	1.0^{6}0 40000005
.0^{3}3	1.0010020 04678364	1.0006010 82381717	.0^{6}1	1.0^{6}0 33333336	1.0^{6}0 20000001
.0^{3}2	1.0006675 56940580	1.0004004 80705128			
.0^{3}1	1.0003335 55728539	1.0002001 20088070	.0^{7}9	1.0^{7} 30000002	1.0^{7} 18000001

Example 3. To find the cube root of 6.7304592 1819061. This when multiplied by .16 gives 1.0768734-74910498. The next multiplier is obviously 1-.07. Before applying it to the first eight digits we observe that the first ten will be affected. Hence we make a mental note of the ninth and tenth, namely 74. After subtracting .010768734 seven times we shift the carriage, put on the keys 749105 and subtract this seven times from the last six digits. At each stage, if we carry in mind a group of not more than four digits from one part of the subtraction to the other, we can effect all the factoring multiplications without any writing or shifting of columns. After the preliminary .16 we get the factors indicated by the digits 07149063 4841112. Only five of these are taken from the table. The last is 1.0000000 11613704, found by dividing the figures 34841112 by 3. The required cube root is 1.8880561 2016121. It is convenient here to use 16-figure factors while taking the original number and the root to 15 figures.

V. A New Calculation of π

By C. C. Camp, University of Illinois

In the May number of the Monthly (*1925*, 253) Professor Bennett gave two hitherto unpublished equations for $\pi/4$. That article is partly responsible for the calculation which follows although my formula was arrived at in January, 1924. Several other formulas were then derived, including his form (8), p. 254, which did not seem at that time to be a sufficient improvement over those used by Rutherford and Machin. In any such calculation one must consider not only the total number of terms necessary for a given number of decimals but also the ease in computing.

Machin's formula requires about .93 n terms where n is the number of decimals employed. The formula[1] used in the present calculation, namely

$$(\pi/4) = 8 \arctan(1/10) - \arctan(1/239) - 4 \arctan(1/515)$$

requires about .898 n terms while the forms given by Professor Bennett require about 1.125 n and 1.118 n terms, respectively.

As for the numerical calculation with Gregory's series, none could be easier than that for arctan (1/10) and the others are readily computed with a Monroe calculator.

The labor may be lessened by using the coefficients as numerators in the several terms to avoid later multiplication. One might combine terms before doing the divisions by 3,5,7, etc. In any case for the last two arctangents it is best to divide by 239^2 or 515^2 successively on the machine, and one may well write alternate terms on different sheets to separate the positive and negative terms. The maximum error was estimated as 12 in the 56th decimal place but the actual error in $\pi/4$ was zero there. The series used were $(8/10 - 8/3 \cdot 10^3 + 8/5 \cdot 10^5 - \cdots) - (1/239 - 1/3 \cdot 239^3 + 1/5 \cdot 239^5 - \cdots) - (4/515 - 4/3 \cdot 515^3 + 4/5 \cdot 515^5 - \cdots)$ and the several terms are exhibited below:

[1] A sketch of the history of this formula is given by H. J. Heyman in *Archivio di Storia della Scienza*, vol. 6 (1925), p. 113. The earliest known reference to it is in a manuscript by S. Klingenstierna (1698–1765) which has recently been found at Upsala. The manuscript is dated "Londini d. 7 Aprilis 1730." Editor.

8/10	.8					
$8/5 \cdot 10^{5}$	.000016					
$8/9 \cdot 10^{9}$	.000000000888888888888	8888888888	8888888888	8888888888	888889−
$8/13 \cdot 10^{13}$	.000000000000006153846	1538461538	4615384615	3846153846	153846
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$8/45 \cdot 10^{45}$	.00 ...			017777	777778−
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$8/53 \cdot 10^{53}$	.00 ...				0151−
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$1/15 \cdot 239^{15}$	.00 ...		01405	8540188257	236882−
$1/19 \cdot 239^{19}$	.00 ...			03401	623797−
$1/23 \cdot 239^{23}$	.00 ...				01−
$4/3 \cdot 515^{3}$	.000000009761511033 10	0368771583	9973448689	9899669969	412915
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Total	.800016035067053255 50	7465581417	1216940390	7584283444	679417
	Negative Terms				
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$8/15 \cdot 10^{15}$	.00 ... 053333	3333333333	3333333333	3333333333	333333
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$8/23 \cdot 10^{23}$	.00 ...	03478260	8695652173	9130434782	608696−
$8/27 \cdot 10^{27}$	.00 ...	0296	2962962962	9629629629	629630−
$8/31 \cdot 10^{31}$	.00 ...		0258064516	1290322580	645161
$8/35 \cdot 10^{35}$	.00 ...		022857	1428571428	571429−
$8/39 \cdot 10^{39}$	.00 ...		02	0512820512	820513−
$8/43 \cdot 10^{43}$	.00 ...			01860465	116279
$8/47 \cdot 10^{47}$	.00 ...			0170	212766−
$8/51 \cdot 10^{51}$	.00 ...				015686
$8/55 \cdot 10^{55}$	.00 ...				01
1/239	.004184100418410041 84	1004184100	4184100418	4100418410	041841
$1/5 \cdot 239^{5}$	.00 ...0025647231	4424647365	7052071108	8299523979	598313
$1/9 \cdot 239^{9}$	.00 ...	0043669315	2440391897	4984975066	914976−
$1/13 \cdot 239^{13}$	.00 ...		0092658216	2415509570	671442
$1/17 \cdot 239^{17}$	.00 ...			0217163464	991096−
$1/21 \cdot 239^{21}$	.00 ...				053880−
4/515	.007766990291262135 92	2330097087	3786407766	9902912621	359223
$4/5 \cdot 515^{5}$	.00 ...0002208278	4880234598	0316727246	9383127306	256882
$4/9 \cdot 515^{9}$	.00 ...	0174402	3853155275	3584633996	931836
$4/13 \cdot 515^{13}$	.00 ...		017164	1642376126	520071−
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Total	−.014617871669604945 89	1804735597	2459729897	8349299067	033893
Positive	.800016035067053255 50	7465581417	1216940390	7584283444	679417
Sum, $\pi/4$	.785398163397448309 61	5660845819	8757210492	9234984377	645524
$\pi =$	3.141592653589793238 46	2643383279	5028841971	6939937510	5820975−

RECENT PUBLICATIONS

EDITED BY W. B. CARVER, Cornell University, to whom books and communications should be sent.

REVIEWS.

Einführung in die Determinantentheorie einschliesslich der Fredholmschen Determinanten. By GERHARD KOWALEWSKI. Second, abridged edition. Berlin, Walter de Gruyter & Co., 1925. 298 pages.

The reviewer's pleasure at hearing that a new edition of Kowalewski's admirable book on determinants is available was considerably tempered when he discovered the nature of the revision. The original work stood in a class by itself not because of a particularly noteworthy exposition of the elements of determinants, though that feature was worthy of praise, but because it brought together in one volume a treatment of analogous concepts from three different fields. The ordinary algebraic determinant found a place in mathematics as a device for solving systems of linear algebraic equations. The "infinite" determinant found a place as a device for solving systems of infinitely many linear equations in infinitely many unknowns. The Fredholm determinant found a place as a device for solving linear integral equations. It was highly fitting that these three concepts should be treated in neighboring pages. And considering the comparative newness of the latter two notions at the time of the first edition of Kowalewski's work (1909), the author deserved great praise for his presentation. In view of this the reviewer had hopes of finding in a second edition, after an interval of sixteen years, some advancement along two different lines. First it was to be expected that the subjects treated, which during the sixteen years have been very much alive, should be recast and brought up to date. Second it was at least within reason to hope that some attempt might be made to *unify* the three concepts mentioned by means of a generalization, of which the three cases treated would be special instances. The latter would have been a particularly acceptable and noteworthy contribution.

It is then a disappointment to note that the edition under review is a mere *abridgement* of the first edition. For this, not the author but the publishers are to blame, according to a statement in the brief preface. Under pressure, the author has reduced the number of pages from 540 to 298. Entire chapters which have been eliminated are those which treated of elementary divisors, determinants of infinitely high order, and Hilbert's characteristic functions of a real symmetric kernel.

Of the chapters which remain, the first ten dealing with the pure theory of the algebraic determinants with applications to systems of linear equations appear to be essentially the same as in the original edition.

Chapter XI on resultants and discriminants, reduced from 32 pages to 9, contains a brief but elegant treatment of Sylvester's dialytic method, and an indication of Bezout's form of the resultant for two forms of the same degree n, details being given for $n=2$ and 3.

Chapter XII on linear and quadratic forms contains most of the material of the corresponding chapter in the first edition, and is particularly pleasing.

Chapter XIII on functional determinants contains a definition of these determinants and a proof of their law of multiplication together with certain applications of this law in about 6 pages. Another 6 pages are devoted to a detailed proof that under suitable restrictions the Jacobian can be obtained as a *limit* of a quotient of two determinants, the elements of the denominator

$$\frac{d(u_1, \cdots u_n)}{d(x_1, \cdots x_n)}$$

being suitably restricted increments of the independent variables x_i, and the elements of the numerator being corresponding increments of the functions u_i. The remaining 12 pages of the chapter have to do with systems of functions from the function-theoretic point of view. The proofs depend largely upon the theory of point sets, and bear little relation to the remainder of the book. In the interest of the much desired condensation, this material might have been replaced by a bare statement of facts without serious loss. All the proofs are easily available elsewhere in at least as satisfactory form.

Chapter XIV on Wronskian and Gramian determinants contains much of the material of the first edition on linear dependence of functions. The notion of a normalized orthogonal system is presented in an elegant fashion. It is to be regretted that space did not permit the inclusion here of the elements of the geometry of function space which the author has developed in memoirs which unfortunately are inaccessible to many students.

Chapter XV on some geometric applications of determinants contains the material of the first edition with the exception of the geometric interpretations of the Jacobian.

Chapter XVI on linear integral equations is apparently identical with the corresponding chapter in the original.

The second edition like the first is remarkably free from errors. A few discrepancies have arisen from the abridgement. For example the chapter under the heading *Resultants and Discriminants* contains no reference to the latter. Again the reference in the final chapter to von Koch's theory of infinite determinants is of little value since the chapter on infinite determinants has been eliminated.

Notwithstanding our regret that the revision is an abridgement, it must be confessed that the book fits its title quite as well as it did before, and it will be a very valuable aid to students.

Of special interest to Americans is the fact that the dedication is to Professor Alexander Ziwet.

L. L. DINES

Lebesguesche Integrale und Fouriersche Reihen. By L. SCHLESINGER AND A. PLESSNER. Berlin, Walter de Gruyter & Co., 1926. viii+229 pages.

The present book is the outcome of a course of lectures given by Professor Schlesinger at Giessen in 1921–2. It is intended to be an introductory treatment of the theory of Lebesgue integration, and is consequently more restricted and more elementary than the treatises of Hausdorff, Carathéodory, and Hahn. In point of difficulty, it stands between these and de la Vallée-Poussin's monograph in the Borel series. The book is divided into six chapters, whose titles are self-explanatory: The fundamental concepts of the theory of sets, the measure of sets of points, functions of real variables, the Lebesgue integral, functions of one and of two variables, Fourier series. The last of these is inserted as an illustration of the importance of the Lebesgue theory in the investigation of a classic problem of analysis. The entire book is thorough, accurate, readable, and well documented.

Since the presentation of the subject-matter does not depart materially from the standard treatments, it is unnecessary to make much detailed comment. It seems to be in place to note certain omissions which are due to the tastes and aims of the authors. Apparently the authors are not disturbed by the battles being waged over the Zermelo principle: they allow it to do its work unseen. They pass over Baire's classification of functions very lightly, no doubt because they propose to focus attention on the theory of integration. Nothing is said concerning the existence or the construction of non-measurable sets. It seems to the reviewer that a few pages devoted to these subjects would provide illumination and stimulation for the student without leading him too far afield.

Typographically the book has been well prepared. Such minor errata as the reviewer noted can be corrected easily as one reads.

M. H. STONE

ARTICLES IN CURRENT PERIODICALS.

The lists appearing regularly in the Monthly of articles in current periodicals are intended to include (1) title of papers in all mathematical journals published in the United States; (2) titles of mathematical papers and reports published by the national and state academies of science and in journals devoted to general science; (3) titles of mathematical papers by American authors published in foreign journals.

Acta Mathematica, volume 47, no. 4: "An extension of Poincaré's last geometric theorem" by George D. Birkhoff, 297-311; "Functions of a complex variable with assigned derivatives at an infinite number of points, and an analogue of Mittag-Leffler's theorem" by Philip Franklin, 371-385.

Bulletin of the American Mathematical Society, volume 32, no. 1, January-February, 1926: "The Heaviside operational calculus" by J. R. Carson, 43-68; "Note on a fundamental lemma concerning the limit of a sum" by H. J. Ettlinger, 69-70; "Note on rational plane cubics" by C. A. Nelson, 71-76; "A trivial Tauberian theorem" by W. A. Hurwitz, 77-82.

Journal of Mathematics and Physics, Massachusetts Institute of Technology, volume 5, no. 3, March, 1926: "The harmonic analysis of irregular motion (Second paper)" by Norbert Wiener, 158-189.

Mathematische Annalen, volume 95, no. 4, February, 1926; "The operational calculus" by Norbert Wiener, 557-584.

Rendiconti del Circolo Mathematico di Palermo, volume 49, no. 2, May-August, 1925: "Trigonometric realms of rationality" by H. Hancock, 263-276; "Asymptotic satellites near the straight line equilibrium points (elliptical case)" by D. Buchanan, 299-304.

UNDERGRADUATE MATHEMATICS CLUBS

All reports of club activities should be sent to H. J. Ettlinger, 3110 Harris Park Ave., Austin, Texas.

CLUB ACTIVITIES

The Mathematics Club of Cooper Union, New York City

The officers for the year 1925-1926 were: Fred Miller, '26, president; Thaddeus Slonczewski, '26, vice president; Walter Judson, '26, secretary and treasurer. Meetings were held at intervals of three or four weeks throughout the year, with the following program:

Nov. 11, 1925. "The use of mathematics in the inspection of gauges and standards" by Mr. Charles H. Lehmann, telephone engineer.

Dec. 9. "The theory and use of the planimeter, with an exhibition of instruments" by Bert Speier '26.

Jan. 6, 1926. "Methods of constructing magic squares" by Fred Mertens '29.

Jan. 27. "Constructions with straight-edge only; constructions with compasses only" by Samuel Lubkin '27.

Feb. 17. "Diophantine equations" by Thaddeus Slonczewski '26.

Mar. 17. "New uses of the slide-rule" by Professor W. E. Breckenridge.

Apr. 7. "The object of a mathematics club" by Professor H. W. Reddick. Discussion of plans for next year, led by Professor W. J. Pickett. Election of officers.

(Report by Mr. Judson)

The Matoon Mathematics Club of Park College, Parkville, Missouri.

The officers of the Club for the year 1925–1926 were: Ora Gates, president; Edna Buckley, vice president; Gladys McClave, secretary and treasurer.

The Club was organized primarily for those who have chosen mathematics as either a major or minor. The attendance at all meetings was very good and a greater interest in mathematics has resulted.

The program for the first semester of the year 1925–1926 was the following:

October 8. "The fourth dimension" by Margaret Creegler.

October 22. "De Moivre's theorem" by Professor R. A. Wells.

November 12. "History of geometry" by Marian Brown.

November 19. "Proofs of Pythagorean theorem" by John Waterman.

December 10. "Development of formula for finding length of sides of a right triangle" by Herma Hudson.

January 14. "Modern views of space" by Professor R. A. Wells.

(Report by Miss Gladys McClave, secretary)

The Mathematical Club of Rutgers University, New Brunswick, New Jersey

The following papers were read before the Club during the academic year 1925–1926:

"The anharmonic ratio" by Professor Richard Morris.

"Parallel lines" by Raymond J. Seeger.

"The Brachistochrone" by Professor S. E. Brasefield.

"Sturm's functions" by Joseph P. Bogdan.

"The cubic equation" by Professor W. V. N. Garretson.

"Cross ratio" by Frank W. Malsbury.

"Trilinear coordinates" by Professor Richard Morris.

"Paradoxes of probability" by R. J. Seeger.

"Nomography" by Herman B. Dresser.

"Relativity" by Howard B. Waxwood, Jr.

"The rectification of the ellipse" by Professor W. E. Breazeale.

"Archimedes' spiral" by R. J. Seeger.

(Report by Professor Morris)

Pi Mu Epsilon, Bucknell University, Lewisburg, Pennsylvania

Officers for the year 1925–1926: Director, Prof. H. S. Everett; Vice-Director, W. I. Miller, '26; Secretary, M. Pauline Lindley, '26; Treasurer, P. C. Wallace, '26; Librarian, Katheryne E. Miller, '26.

Program for the year 1925–1926:

Oct. 26. Business meeting, election to membership.

Nov. 11. Annual initiation and banquet.

Dec. 17. "Groups, an exposition of the field" by Prof. H. S. Everett.

Jan. 21. "Our pioneer mathematics professors, who they were and what they did" by Prof. W. C. Bartol, head of the department for forty-seven years.

Feb. 24. "The origin of certain mathematical symbols" by Miss Katherine Gaventa, '27.

Mar. 25. "Atomic structure" by Professor V. B. Hall.

May 26. Annual picnic and election of officers. At this meeting a prize (slide rule) was voted that member of the integral calculus classes who attained the highest standing in regular freshman and sophomore mathematics. Awarded to V. W. McHail, '28.

The following officers were elected for 1926–1927: Director, Prof. H. S. Everett; Vice-Director, Christopher Mathewson, Jr., '27; Secretary, Agnes Dunbar, '27; Treasurer, K. W. Horsman, '27; Librarian, Evelyn Deen, '27.

The Mathematics Club of the New Jersey College for Women, New Brunswick, New Jersey

During the year 1925–1926, the following papers were read:

"Continued fractions" by Lois Schenck.

"The relation of mathematics to English" by Henrian Emerson.

"Benjamin Peirce" by Dorothy Brown.

"The Pythagorean theorem" by Vera Joslin.

"The cycloid" by Elizabeth Baier.
"Pythagoras" by Ruth Palmer.
"A problem in analytic geometry" by Professor E. P. Starke.
"Bernoullian numbers" by Ruth Thompson.
"Remarkable calculators" by Emily Mott.
"Calculus of variations" by R. J. Seeger.
"The origin of mathematics" by Muriel Blackford.
"Anharmonic ratios" by Professor Richard Morris.
"Indeterminate equations" by Ruth Ruegg.
"Triangles and circles" by Professor Richard Morris.
"Work on the United States Coast and Geodetic Survey" by Professor A. A. Titsworth.
"Egyptian mathematics" by Anna Pokorny.

(Report by Professor Morris)

Pi Mu Epsilon, Hunter College, New York City

The Hunter College Chapter of Pi Mu Epsilon has had a successful year. The first semester wa. devoted to the study of relativity. Each member of the chapter was required to do certain reading on the subject. Reports and talks were made at the meeting by individuals. Meetings were held monthlys More frequent meetings were not held for the reason that we did not wish to interfere with Mathematics Club, a larger organization open to all and to which all the members of Pi Mu Epsilon belong. For this same reason, Pi Mu Epsilon at Hunter College has made of itself primarily an organization for study rather than for the delivery of papers.

At the midwinter dinner of the American Mathematical Society, certain members of the chapter presented a playlet entitled "A Drama in Relativity." For this entertainment they received the officia thanks of the Society.

The highwater mark of the year's accomplishment was reached when in February, Professor Edward Kasner of Columbia University was initiated as an active member of the Hunter College Chapter. At this time, the following undergraduates were also initiated: Misses Esther Gurwitz, Esther Lacher, Lillian Parker and Sylvia Sider.

The second semester was devoted to the study of statistics. The meetings were held as in the fall.

The chapter is again offering a ten dollar gold piece as a prize to that student in the graduating class at Hunter College High School, who presents the best record in mathematics.

The officials for the year 1925–1926 were: Director, Professor Tomlinson Fort; Vice-Director Rosalind Honig; Secretary, Agnes Corcoran; Treasurer, Mary Draper.

(Report by Professor Tomlinson Fort)

The Mathematics Club of Wellesley College, Wellesley, Massachusetts

The officers for the year 1925–1926 were: President, Elizabeth Maxon, '26; vice president, Nina Hammond, '26; faculty executive member, Miss Lennie Copeland; senior executive member, Charlotte Banta, '26; junior executive member, Dorothy Graef, '27; secretary and treasurer, Blanche Weatherhead, '27.

The following meetings were held.

October 16. Business meeting. Game-problem contest.

November 10. "Use of alignment charts" by Ruth Mason, '26 and Margaret Lane, '26. "Graphs used in statistics" by Helen Sawin, '27 and Frances Baume, '27. "Semi logarithmic and logarithmic graphs" by Miriam Dice, '27 and Grace Loveland, '27.

December 11. Display of mathematical instruments and models.

February 12. Lecture by Mr. A. Harry Wheeler of Worcester, Massachusetts, "An esthetic side of geometry."

March 19. "The mathematics of life insurance" by Margaret Fairbanks, '27. "The mathematics of navigation" by Isabel McKerracher, '27. "The mathematics of commerce" by Emily Frame, '26. "The mathematics of religion and philosophy" by Eleanor Loomis, '26.

April 16. "The mathematics of the physiologist and physician" by Ruth Foljambe, '27. "The mathematics of biology" by Dorothy Beaton, '27. "The mathematics of art and architecture" by Clara Meade, '26. "The mathematics of domestic arts" by Amy Kenny, '27.

May 14. Election of officers. Supper.

(Report by Miss Blanche Weatherhead, secretary)

PROBLEMS AND SOLUTIONS

EDITED BY B. F. FINKEL, OTTO DUNKEL, AND H. L. OLSON.

Send all communications about Problems and Solutions to B. F. Finkel, Springfield, Mo. All manuscripts should be typewritten, with double spacing and with a margin at least one inch wide on the left.

PROBLEMS FOR SOLUTION

N. B. Problems containing results believed to be new, or extensions of old results are especially sought. The editorial work would be greatly facilitated if, on sending in problems, proposers would also enclose any solutions or information that will assist the editors in checking the statements. In general, problems in well-known textbooks, or results found in readily accessible sources, will not be proposed as problems for solution in the MONTHLY. In so far as possible, however, the editors will be glad to assist members of the Association with their difficulties in the solution of such problems.

3217. Proposed by R. H. Sciobereti, Berkeley, California.

Find a curve such that the radius of curvature at any point is proportional to the reciprocal of the normal.

3218. Proposed by F. M. Garnett, Savannah, Georgia.

A rectangular lighter $a=40$ ft. by $b=20$ ft. moves upstream due west, at the uniform speed $v_1=8$ miles per hour. What time will be required for a swimmer who begins at the southeast corner of the lighter to swim around it while it is in motion, if his rate upstream $v_2=10$ miles per hour, and downstream $v_3=16$ miles per hour?

3219. Proposed by Philip Fitch, Denver, Colorado.

Construct a polygon similar to a given polygon and having the reciprocal of its area equal to the sum of the reciprocals of the areas of a certain number of given polygons.

3220. Proposed by W. L. Ayres, University of Pennsylvania.

Find the value of

$$2^n-(n-1)2^{n-2}+\frac{(n-2)(n-3)}{2!}2^{n-4}-\frac{(n-3)(n-4)(n-5)}{3!}2^{n-6}+\cdots$$
$$+(-1)^{r+1}\frac{(n-r+1)(n-r)\cdots(n-2r+3)}{(r-1)!}2^{n-2r+2}+\cdots,$$

where the number of terms is the greatest integer in $(n+2)/2$ and n is a non-negative integer.

3221. Proposed by H. E. Trefethen, Colby College.

A variable rectangle has a diagonal of constant length and two sides lying upon two fixed perpendicular straight lines. Determine geometrically the locus of the foot of the perpendicular from the vertex opposite the fixed vertex upon the diagonal which does not pass through that vertex.

3222. Proposed by Norman Anning, University of Michigan.

Under what conditions is it possible to choose five points in space such that the straight line joining any two shall be perpendicular to the plane which contains the remaining three?

3223. Proposed by Paul Capron, U. S. Naval Academy.

A circle of radius b and a straight line at a distance a from the center of the circle, lie in the same plane; the circle is revolved about the line generating a torus. A plane Π is passed through the axis l, intersecting the torus in two circles S_1 and S_2; a plane Σ is passed perpendicular to Π and containing a common interior tangent of S_1 and S_2. Show that Σ intersects the torus in two circles.

3224. Proposed by Nathan Altshiller-Court, University of Oklahoma.

Find the locus of the center of gravity of the variable triangle determined by three skew lines in a plane turning about a fixed axis.

3225. Proposed by C. N. Schmall, New York, N.Y.

If $ABCD$ is a cyclic quadrilateral, and P any point in its plane prove that

$$PA^2 \cdot \Delta BCD + PC^2 \cdot \Delta ABD = PB^2 \cdot \Delta ACD + PD^2 \cdot \Delta ABC.$$

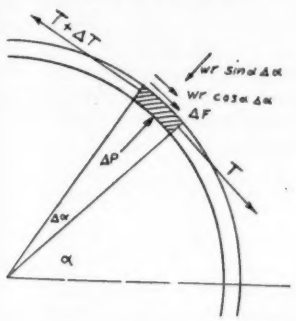
SOLUTIONS

3139 [1925, 261]. Proposed by C. N. Mills, Illinois State Normal University.

A rope weighs w pounds per foot. How many coils of the rope must be taken around a rough cylinder of radius r in order that a weight may support another n times as great, k being the coefficient of friction between the rope and the cylinder.

SOLUTION BY MICHAEL GOLDBERG, Washington, D. C.

From the figure it is seen that



$$\Delta T = \Delta F + wr\cos\alpha\,\Delta\alpha,$$
$$= k\Delta P + wr\cos\alpha\,\Delta\alpha. \tag{1}$$

$$(T+\Delta T+T)\sin(\Delta\alpha/2) + wr\sin\alpha\,\Delta\alpha = \Delta P. \tag{2}$$

If we eliminate ΔP from (1) and (2) and pass to the limit, there results the differential equation

$$dT/d\alpha - kT = wrk\sin\alpha + wr\cos\alpha \tag{3}$$

whose complete integral is

$$T = \frac{wr}{(1+k^2)}\left[(1-k^2)\sin\alpha - 2k\cos\alpha\right] + Ce^{k\alpha}. \tag{4}$$

When $\alpha=\alpha_1=0$, $T=T_1=C-2wrk/(1+k^2)$.
When $\alpha=\alpha_2$, $T=T_2=nT_1$.
Equation (4) can now be written

$$\frac{e^{k\alpha_2}}{1+k^2} = \frac{nT_1 - wr\sin(\alpha_2-\beta)}{T_1(1+k^2)+2wrk}, \tag{5}$$

where $\sin\beta = 2k/(1+k^2)$, $\cos\beta = (1-k^2)/(1+k^2)$; α_2 must then be found by trial from (5).

3145 [1925, 385]. Proposed by W. J. Sidis, New York City.

Prove that if a number of n digits, expressed in the scale of r, is divisible by any factor of r^n-1, that divisibility is not altered by a cyclical permutation of the digits of the original number.

SOLUTION BY ORMOND STONE, Clifton Station, Va.

Let $\overline{\alpha_1 \cdots \alpha_n}$ be the number referred to in which $\alpha_1, \cdots, \alpha_n$ are the n digits. Then

$$\overline{\alpha_2 \cdots \alpha_n\alpha_1} = \overline{\alpha_2 \cdots \alpha_n}\cdot r + \alpha_1$$
$$= \overline{\alpha_1 \cdots \alpha_n}\cdot r - \alpha_1(r^n-1),$$

from which the proposition follows immediately.

Also solved by HARRY LANGMAN, L. C. MATHEWSON, and A. PELLETIER.

3146 [1925, 385]. Proposed by L. H. Burns, Student, Yale University.

Show, by elementary geometrical methods (preferably suitable for use in teaching a class in elementary solid geometry), that the volume of a regular icosahedron of edge a is

$$5a^3(3+\sqrt{5})/12,$$

and obtain a similar formula for the volume of a regular dodecahedron of edge a.

SOLUTION BY N. PETROFF, Student, Cooper Union.

The following formulae for regular polygons of side a will be used:

Area of equilateral triangle, $a^2\sqrt{3}/4$ (1)

Radius of circumscribed circle, $a/\sqrt{3}$ (2)

Area of regular pentagon, $a^2\sqrt{25+10\sqrt{5}}/4$ (3)

Radius of circumscribed circle, $a\sqrt{5+\sqrt{5}}/\sqrt{10}$ (4)

Length of diagonal, $a(1+\sqrt{5})/2$ (5)

A. Regular Icosahedron. Consider five faces having V for a common vertex. The five bases of these faces form a regular pentagon of side a, and the perpendicular from V to this pentagon passes through its center C and the opposite vertex V' of the icosahedron. Let A be a vertex of the pentagon, R_i and r_i the radii of the circumscribed and inscribed spheres, then VAV' is a right angle. Hence $VA^2=VV'.VC$, or $a^2=2R_i\sqrt{a^2-CA^2}$. Inserting the value of CA given by (4) we have after reduction

$$R_i = \frac{a}{4}\sqrt{10+2\sqrt{5}}. \tag{6}$$

Now consider one of the pyramids whose base is a face and whose lateral edges are radii R_i. The length of the altitude is r_i and its foot is the center of the equilateral triangle which forms the base. Hence r_i, the length given by (2), and R_i form a right triangle. Using (6), we obtain from this right triangle

$$r_i = \frac{a}{2}\sqrt{\frac{7+3\sqrt{5}}{6}}. \tag{7}$$

The volume of the solid is now obtained from (7) and (1)

$$V_i = \frac{5a^3}{12}(3+\sqrt{5}). \tag{8}$$

B. Regular Dodecahedron. We may use the same analysis as before. Here there are three faces having a common vertex at V which is the vertex of a pyramid whose base is an equilateral triangle of side given by (5). Here CA is given by (2) and (5). After a reduction, we find

$$R_d = a(\sqrt{3}+\sqrt{15})/4,\ r_d = a\sqrt{\frac{25+11\sqrt{5}}{40}},\ V_d = \frac{a^3}{4}(15+7\sqrt{5}).$$

NOTE BY OTTO DUNKEL. If the middle points of the faces of a regular polyhedron P be joined by segments of straight lines, we obtain a second regular polyhedron. Let P' be a polyhedron similar to this second one. Let R and r denote the radii of the circumscribed and inscribed spheres of P, and let a, ρ, b denote the length of a side of a face, the radius of the circumscribed circle and the apothem. Also let the same letters accented denote the corresponding parts of P'. Then from similar triangles of a figure, it will be easily seen that

$$\frac{r}{\rho} = \frac{r'}{\rho'} = \frac{1}{\sqrt{\left(\frac{aa'}{4bb'}\right)^2 - 1}}. \tag{9}$$

If P is an icosahedron, then P' is a dodecahedron and the ratio in (9) has the value $(3+\sqrt{5})/4$. This result may be used to compute the volumes of the two solids.

Also solved by HARRY LANGMAN, A. PELLETIER, S. D. TURNER, and the PROPOSER.

3149 [3145; 1925, 433; 1926, 389]. Proposed by W. D. Cairns, Oberlin, Ohio.

The center of gravity of any zone of a certain surface of revolution lies midway between the bases of the zone. What is the surface?

A solution of this problem bearing the incorrect number 3145 appeared in the August-September number. Also solved by MICHAEL GOLDBERG, A. PELLETIER, J. B. REYNOLDS, and the PROPOSER.

3150 [3146; 1925, 433]. Proposed by W. C. Eells, Whitman College.

For what integral values of s and n is it true that the nth power of any integer leaves the same remainder when divided by s as does the integer itself when divided by s? (Generalized from problem 873, *School Science and Mathematics*, March 1925).

SOLUTION BY H. L. OLSON, Michigan State College.

It is evidently assumed that n is a positive integer. First, let s be a prime number; we must have x^n-x divisible by s, for all integers x. Hence either x or $x^{n-1}-1$ must be divisible by s. According to Fermat's theorem, if x is not divisible by s, then $x^{s-1}-1$ is divisible by s. Since there are numbers x (primitive roots) such that no lower power of x than x^{s-1} will leave a remainder 1 when divided by s, it follows that $n-1$ must be a multiple of $s-1$, i.e., $n=k(s-1)+1$, where k is a positive integer or 0.

Next let $s=s_1s_2\cdots s_t$, a product of distinct prime numbers; then x^n-x must be divisible separately by $s_1, s_2\cdots s_t$, for all integers x. Hence $n-1$ must be divisible by $s_1-1, s_2-1, \cdots, s_t-1$, and therefore by their least common multiple, m; *i.e.* $n=km+1$, where k is a positive integer or 0.

Finally, let $s=s_1^{\epsilon_1}s_2^{\epsilon_2}\cdots s_t^{\epsilon_t}$, where at least one of the exponents, say $\epsilon_r, >1$. If x is divisible by s_r, $x^{n-1}-1$ is prime to s_r, and hence, unless x is divisible also by $s_r^{\epsilon_r}$, x^n-x is not divisible by $s_r^{\epsilon_r}$. Hence no number s of this form satisfies the conditions of the problem.

Also solved by W. E. ROTH.

NOTES AND NEWS

Readers are invited to contribute to the general interest of this department by sending items to H. W. Kuhn, Ohio State University, Columbus, Ohio.

Dr. TOBIAS DANTZIG has been appointed assistant professor of mathematics and mechanics at the University of Maryland.

Professor J. E. DAVIS, of Pennsylvania State College, has been appointed associate professor of mathematics at Drexel Institute.

Professor W. C. EELLS, of Whitman College, has been granted leave of absence for the academic year 1926–27 to serve as research assistant in educational measurements at Stanford University.

Mr. IRA GWINN has been appointed to a professorship at Morningside College.

Dr. R. W. HARTLEY has been appointed professor of mathematics at Southwestern College, Memphis.

Dr. G. W. HESS has been appointed associate professor of mathematics at Howard College, Birmingham.

Dr. K. W. LAMSON, of Columbia University, has been appointed assistant professor of mathematics at Lehigh University.

Dr. B. Z. LINFIELD has been appointed associate professor of mathematics at the University of Virginia.

Dr. W. L. MACHMER, of the department of mathematics at the Massachusetts Agricultural College, has been made dean of that college.

Dr. B. C. PATTERSON has been appointed assistant professor of mathematics at Washington and Jefferson College.

Professor GEORGIA E. ROBINSON, of Shorter College, has been appointed professor of mathematics at Jefferson City Junior College.

Mr. H. A. ROBINSON has been appointed head of the department of mathematics at Agnes Scott College.

Miss JESSIE M. SHORT has been promoted to an assistant professorship of mathematics at Reed College.

Assistant Professor J. K. WHITTEMORE has been promoted to an associate professorship of mathematics at Yale University.

At the University of Michigan, W. M. COATES, instructor in mathematics, has returned after two years' study of aeronautics in Germany and Austria. The following new instructors have been appointed: L. W. COHEN, R. H. MARQUIS, M. J. THOMPSON, J. H. BUSHEY, W. R. JONES.

A limited number of fellowships for advanced study in various subjects, including mathematics, at French Universities will be awarded for the 1927–28 by the Institute of International Education. Each will carry a stipend of $1200 and will be tenable for one year, with possible renewal if circumstances are favorable. Applications must be received not later than January 1, 1927. Full information and application blanks may be obtained from the executive secretary, STEPHEN P. DUGGAN, 522 Fifth Avenue, New York City.

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DIRECTORY

EDITORIAL CORRESPONDENCE should be addressed to the EDITOR-IN-CHIEF. W. B. FORD, 204 Mason Hall, Ann Arbor, Mich.

BOOKS FOR REVIEW should be sent to W. B. CARVER, White Hall, Ithaca, N. Y.

BUSINESS CORRESPONDENCE should be addressed to the SECRETARY-TREASURER of the Association, W. D. CAIRNS, Oberlin, Ohio.

MEETINGS OF THE ASSOCIATION AND ITS SECTIONS

Eleventh Annual Meeting, Philadelphia, Pa., December, 30-31, 1926.

The following are dates of Section Meetings of the Association in 1926:

ILLINOIS, Decatur, Ill., May 7-8.

INDIANA, Purdue University, May, 7–8.

IOWA, Cedar Rapids, April.

KANSAS, Merged in National Meeting.

KENTUCKY, Berea College, May 1.

LOUISIANA-MISSISSIPPI, New Orleans, La., March 12-13.

MARYLAND - DISTRICT OF COLUMBIA - VIRGINIA, Annapolis, Md., December 4.

MICHIGAN, Ann Arbor, Mich., April 1.

MINNESOTA, Northfield, Minn., May 22.

MISSOURI, Kansas City, Mo., November.

NEBRASKA, Bethany, Neb., May.

OHIO, Columbus, Ohio, April 2.

ROCKY MOUNTAIN, Colorado College, April, 1927.

SOUTHEASTERN, Atlanta, Ga., March 19-20.

SOUTHERN CALIFORNIA, Los Angeles, Calif., November 6.

TEXAS, November.

Secretaries of Sections will please report changes or corrections promptly to the Editor.

A Brief Course in Analytic Geometry
And the Elements of Curve-Fitting

By WALTER B. FORD, *University of Michigan*, with the cooperation of RAYMOND W. BARNARD, *University of Chicago*

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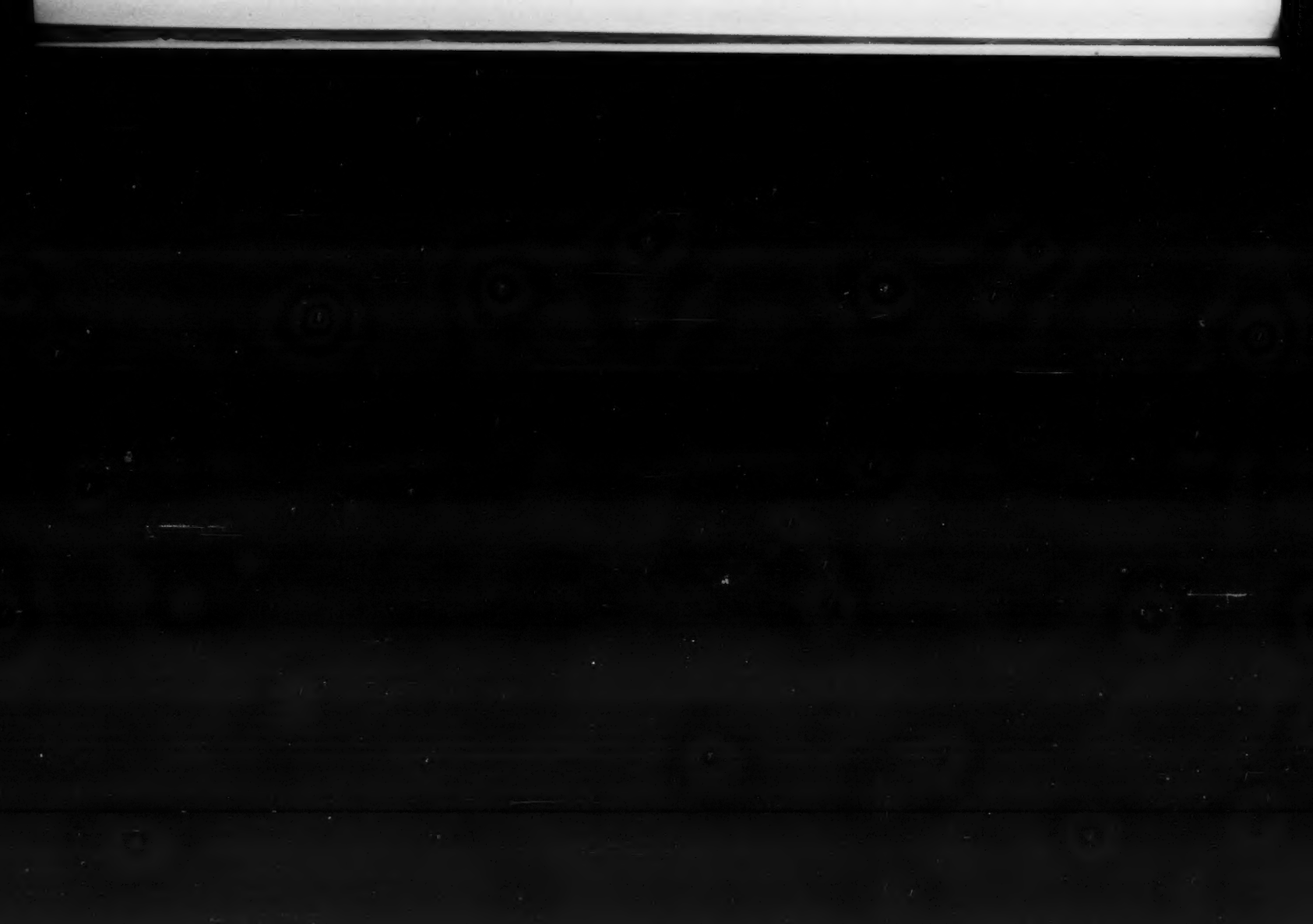

TENTH SUMMER MEETING OF THE ASSOCIATION

The tenth summer meeting of the Mathematical Association of America was held at Ohio State University, Columbus, O., on September 7–8, 1926, preceding and in conjunction with the summer meeting of the American Mathematical Society. The total attendance was 151, including the following members of the Association.

O. P. Akers, Allegheny College.
W. E. Anderson, Miami University.
N. H. Anning, University of Michigan.
C. L. Arnold, Ohio State University.

Gladys L. Banes, Butler University.
Grace M. Bareis, Ohio State University.
I. A. Barnett, University of Cincinnati.
H. M. Beatty, Ohio State University.
Ethelwynn R. Beckwith, Milwaukee-Downer College.
E. T. Bell, California Institute of Technology.
Susan R. Benedict, Smith College.
A. A. Bennett, Lehigh University.
W. W. Bigelow, U. S. Coast and Geodetic Survey.
Henry Blumberg, Ohio State University.
R. L. Borger, Ohio University.
C. F. Bowles, South Dakota State School of Mines.
C. T. Bumer, Ohio State University.
W. H. Bussey, University of Minnesota.

W. D. Cairns, Oberlin College.
C. C. Camp, University of Illinois.
J. W. Campbell, University of Alberta.
A. L. Candy, University of Nebraska.
V. B. Caris, Ohio State University.
R. D. Carmichael, University of Illinois.
E. W. Chittenden, University of Iowa.
E. H. Clarke, Hiram College.
A. Cohen, Johns Hopkins University.
Lennie P. Copeland, Wellesley College.
C. C. Craig, University of Michigan.
W. H. Cramblet, Bethany College.
Rufus Crane, Ohio Wesleyan University.
F. Cumming, Columbia University.

J. M. Davis, University of Kentucky.
S. C. Davisson, Indiana University.
Arnold Dresden, University of Wisconsin.

Arnold Emch, University of Illinois.
G. C. Evans, Rice Institute.
H. S. Everett, Bucknell University.

Fay Farnum, Iowa State College.
Peter Field, University of Michigan.
B. F. Finkel, Drury College.
W. B. Fite, Columbia University.
F. A. Foraker, University of Pittsburgh.
W. B. Ford, University of Michigan.
T. C. Fry, Bell Tel. Laboratories, New York.

C. A. Garabedian, University of Cincinnati.
W. V. N. Garretson, Rutgers University.
D. C. Gillespie, Cornell University.
O. E. Glenn, University of Pennsylvania.
B. C. Glover, Otterbein College.
J. W. Glover, University of Michigan.
C. F. Gummer, Queen's University.

W. M. Hamilton, U. S. Naval Observatory, Washington.
C. T. Hazard, Purdue University.
E. R. Hedrick, University of California, Southern Branch.
T. R. Hollcroft, Wells College.

M. H. Ingraham, Brown University.

Dunham Jackson, University of Minnesota.

A. J. Kempner, University of Colorado.
E. C. Kiefer, James Millikin University.
H. W. Kuhn, Ohio State University.

Anna D. Lewis, Lake Erie College.
F. A. Lewis, University of Alabama.
Flora E. Le Stourgeon, University of Kentucky.
C. A. Lindemann, Bucknell University.

C. C. MacDuffee, Ohio State University.
T. E. Mason, Purdue University.
A. S. Merrill, University of Montana.
A. D. Michal, Princeton University.
J. N. Michie, Texas Technological College.
I. L. Miller, South Dakota State College of A. and M. Arts.
W. I. Miller, University of Pittsburgh.
C. N. Mills, Illinois State Normal University.